MAT124 MATHEMATICS II

Parametric Surfaces and Surface Integrals

Outline

Independence of Path Review

Parametric Surfaces

Smooth Surfaces and Area Elements

Oriented Surfaces and Flux Integrals

Independence of Path Review

Line Integrals of Vector Fields

Independence of Path — Example

EXAMPLE

Evaluate $I = \oint_C \left(e^x \sin y + 3y\right) dx + \left(e^x \cos y + 2x - 2y\right) dy$, where C is the ellipse $4x^2 + y^2 = 4$ oriented counterclockwise.

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Solution: $I = \oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}$ is the vector field

$$\mathbf{F} = (e^x \sin y + 3y)\mathbf{i} + (e^x \cos y + 2x - 2y)\mathbf{j}$$

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$$\Big\downarrow \; C : x = \cos t,\ y = 2\sin t,\ 0 \le t \le 2\pi$$

$$-2\int_0^{2\pi} \sin^2 t\,dt = -2\int_0^{2\pi} \frac{1-\cos(2t)}{2}\,dt = \boxed{-2\pi}.$$

Parametric Surfaces

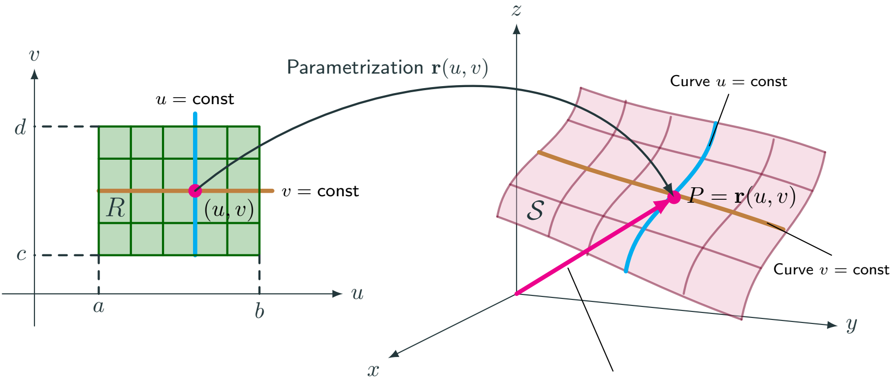
Surfaces and Surface Integrals

Parametric Surfaces

Definition

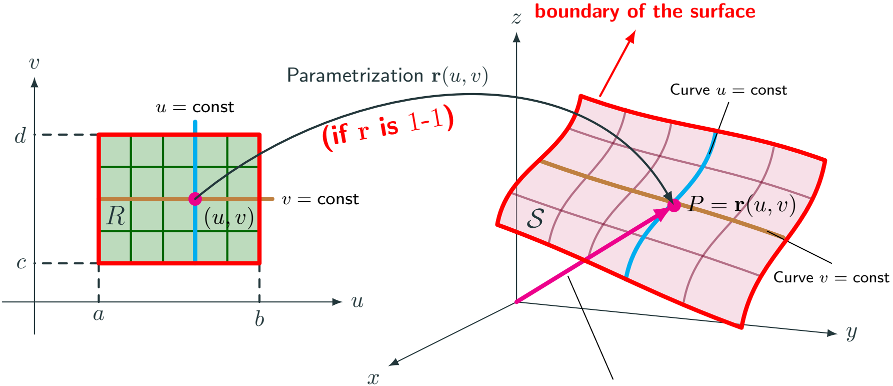
A **parametric surface** in 3-space is a continuous function $\mathbf{r}$ defined on some rectangle R given by $a \le u \le b$, $c \le v \le d$ in the uv-plane and having values in 3-space:

$$\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}, \quad (u,v) \in R.$$



$\mathbf{r}(u,v) = f(u,v)\mathbf{i} + g(u,v)\mathbf{j} + h(u,v)\mathbf{k}$,

Position vector to surface point



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Surfaces and Surface Integrals

Parametric Surfaces

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Describe the surface $\mathbf{r} = a\cos u \sin v\,\mathbf{i} + a\sin u \sin v\,\mathbf{j} + a\cos v\,\mathbf{k}$, where $0 \le u \le 2\pi,\ 0 \le v \le \pi/2$, and $a > 0$. What is its boundary?

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Solution:

$$\left.\begin{aligned} x &= a\cos u\sin v \\ y &= a\sin u\sin v \\ z &= a\cos v \end{aligned}\right] \quad \longrightarrow \quad x^2+y^2+z^2=a^2$$

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Describe the surface $\mathbf{r} = a\cos u \sin v\,\mathbf{i} + a\sin u \sin v\,\mathbf{j} + a\cos v\,\mathbf{k}$, where $0 \le u \le 2\pi$, $0 \le v \le \pi/2$, and $a > 0$. What is its boundary?

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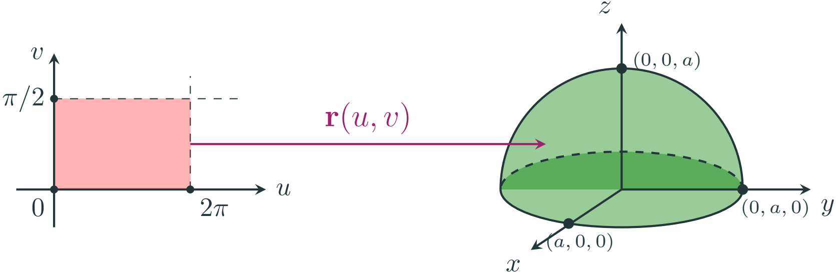

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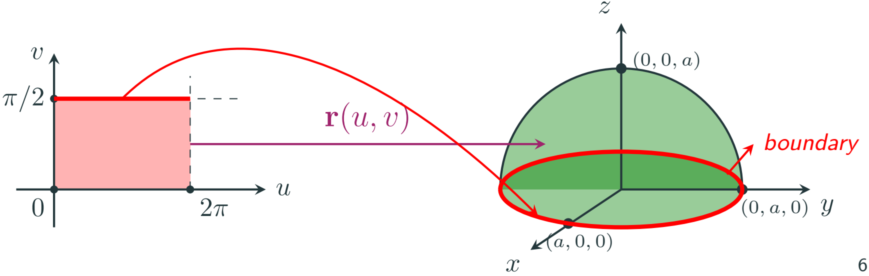

Surfaces and Surface Integrals

Parametric Surfaces

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Describe the surface $\mathbf{r} = a\cos u \sin v\,\mathbf{i} + a\sin u\sin v\,\mathbf{j} + a\cos v\,\mathbf{k}$, where $0 \le u \le 2\pi$, $\boxed{0 \le v \le \pi/2}$, and $\boxed{a > 0}$. What is its boundary?

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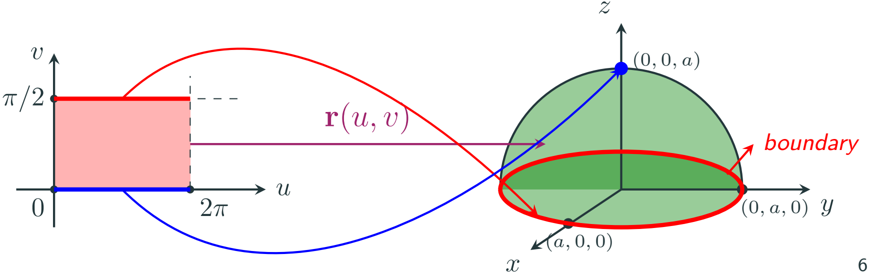

Surfaces and Surface Integrals

Parametric Surfaces

Remark

Like curve parametrizations, surface parametrizations are not unique. Different maps can describe exactly the same geometric surface. For instance, the surface in the preceding example can also be parametrized as follows:

$$\mathbf{r}(u,v) = u\,\mathbf{i} + v\,\mathbf{j} + \sqrt{a^2 - u^2 - v^2}\,\mathbf{k}, \quad u^2 + v^2 \le a^2.$$

Here, the domain of $\mathbf{r}$ is ~~the~~ a closed disk of radius a.

Smooth Surfaces and Area Elements

Surfaces and Surface Integrals

Smooth Surfaces, Normals, and Area Elements

Definition

A set S in 3-space is a **smooth surface** if any point P in S has a neighbourhood N (an open ball of positive radius centred at P) that is the domain of a smooth function $g(x, y, z)$ satisfying:

(i) $N \cap S = \{Q \in N \,:\, g(Q) = 0\}$ and

(ii) $\nabla g(Q) \neq \mathbf{0}$, if Q is in $N \cap S$.

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The part of S lying inside the ball N is a part of a level surface of the smooth function g.

S has a unique tangent plane at any point in N.

Surfaces and Surface Integrals

Smooth Surfaces

For example the cone $x^2 + y^2 = z^2$ with the origin removed is a smooth surface. Not that $\nabla(x^2 + y^2 - z^2) = 0$ at the origin, and so the full cone is not smooth there, since it does not have a unique tangent plane.

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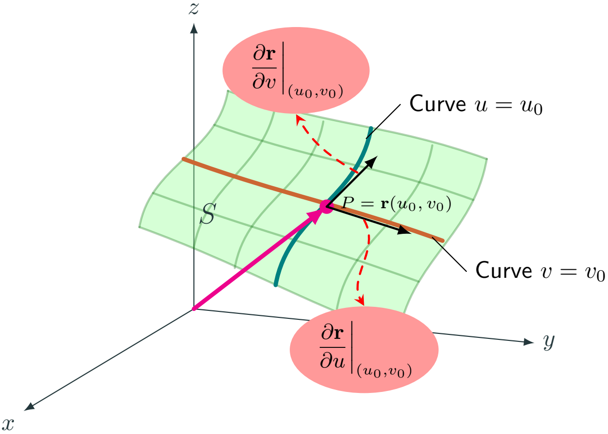
A parametric surface cannot satisfy the condition of the smoothness definition at its boundary points, but will be called smooth if it satisfies the condition at all its nonboundary points.

Surfaces and Surface Integrals

Smooth Surfaces, Normals, and Area Elements



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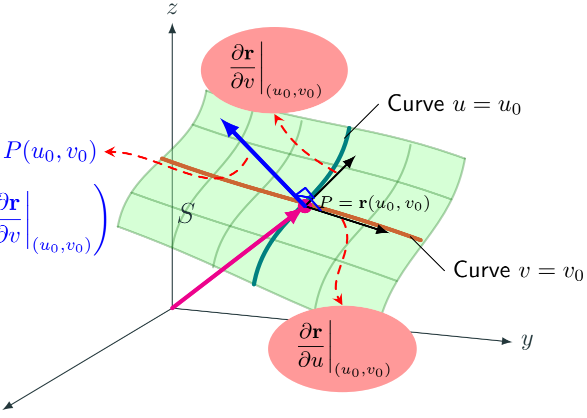
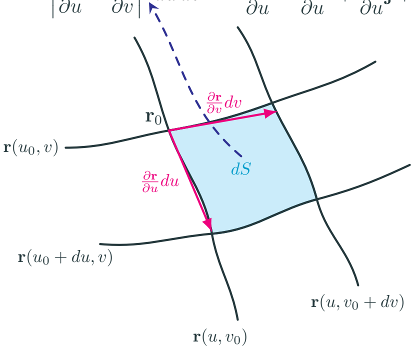


$$\left(\left.\frac{\partial \mathbf{r}}{\partial u}\right|_{(u_0,v_0)} \right) \times \left(\left.\frac{\partial \mathbf{r}}{\partial v}\right|_{(u_0,v_0)} \right)$$

Surfaces and Surface Integrals

Smooth Surfaces, Normals, and Area Elements

$$dS = \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| du\, dv \qquad \frac{\partial \mathbf{r}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}\,, \quad \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$



Surfaces and Surface Integrals

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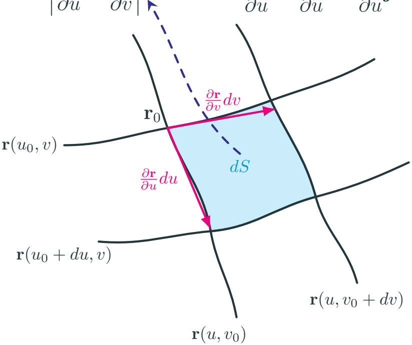

the **normal vector** to S at $\mathbf{r}(u, v)$ is

$$\mathbf{n} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}$$

$$= \frac{\partial(y, z)}{\partial(u, v)}\mathbf{i} + \frac{\partial(z, x)}{\partial(u, v)}\mathbf{j} + \frac{\partial(x, y)}{\partial(u, v)}\mathbf{k}.$$

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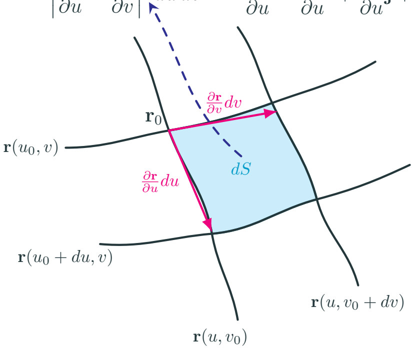

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$$= \frac{\partial(y, z)}{\partial(u, v)}\mathbf{i} + \frac{\partial(z, x)}{\partial(u, v)}\mathbf{j} + \frac{\partial(x, y)}{\partial(u, v)}\mathbf{k}.$$

Also, the **area element** at a point $\mathbf{r}(u, v)$ on the surface is given by

$$dS = \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| du\, dv$$

$$= \sqrt{\left(\frac{\partial(y, z)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(z, x)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(x, y)}{\partial(u, v)}\right)^2}\, du\, dv.$$

Surfaces and Surface Integrals

Smooth Surfaces, Normals, and Area Elements

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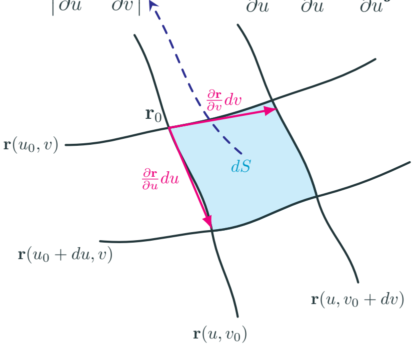

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The area of the surface itself is the “sum” of these area elements:

$$\text{Area of } S = \iint_S dS.$$

Surfaces and Surface Integrals

Smooth Surfaces, Normals, and Area Elements

In general, the surface integral of a function $f(\mathbf{r}) = f(x,y,z)$ over the surface S defined by the parametric equations $\mathbf{r} = \mathbf{r}(u,v)$ for (u,v) in the domain D of the uv-plane is given by

$$\begin{aligned}
\iint_S f\,dS &= \iint_D f\big(\mathbf{r}(u,v)\big)\left\|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right\| du\,dv \\
&= \iint_D f\big(x(u,v), y(u,v), z(u,v)\big) \\
&\qquad \times \sqrt{\left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2}\, du\,dv.
\end{aligned}$$

Surfaces and Surface Integrals

Smooth Surfaces, Normals, and Area Elements

EXAMPLE

The graph $z = g(x, y)$ of a function g with continuous first partial derivative in a domain D of the xy-plane can be regarded as a parametric surface S with the parametrization

$$x = u, \quad y = v, \quad z = g(u, v), \qquad (u, v) \in D.$$

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In this case,

$$\frac{\partial(y, z)}{\partial(u, v)} = -g_1(u, v), \quad \frac{\partial(z, x)}{\partial(u, v)} = -g_2(u, v), \quad \frac{\partial(x, y)}{\partial(u, v)} = 1,$$

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and, since the parameter region coincides with the domain D of g, the surface integral of $f(x, y, z)$ over S can be expressed as a double integral over D:

$$\iint_S f(x,y,z)\,dS = \iint_D f\big(x,y,g(x,y)\big)\sqrt{1 + \big(g_1(x,y)\big)^2 + \big(g_2(x,y)\big)^2}\,dx\,dy.$$

Surfaces and Surface Integrals

Evaluating Surface Integrals

EXAMPLE

Evaluate $\iint_S z\,dS$ over the conical surface $z = \sqrt{x^2 + y^2}$ between $z = 0$ and $z = 1$.

Surfaces and Surface Integrals

Evaluating Surface Integrals

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Solution: Since $z^2 = x^2 + y^2$ on the surface $\mathcal{S}$, we have $\partial z/\partial x = x/z$ and $\partial z/\partial y = y/z$. Therefore,

$$dS = \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}}\,dx\,dy = \sqrt{\frac{z^2+z^2}{z^2}}\,dx\,dy = \sqrt{2}\,dx\,dy.$$

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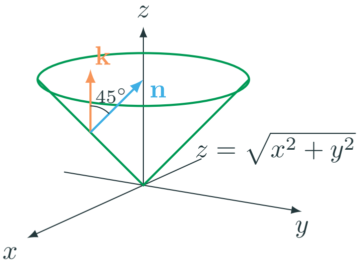

$$\iint_{\mathcal{S}} z\,dS = \sqrt{2}\iint_{x^2+y^2\le 1} z\,dx\,dy$$

$$= \sqrt{2}\int_0^{2\pi} d\theta \int_0^1 r^2\,dr = \frac{2\sqrt{2}\pi}{3}.$$

Surfaces and Surface Integrals

Evaluating Surface Integrals

EXAMPLE

Find $\iint_S (x^2 + y^2)\, dS$, where S is the parametric curve $x = 2uv$, $y = u^2 - v^2$, $z = u^2 + v^2$, $u^2 + v^2 \leq 1$.

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Solution:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2v & 2u \\ 2u & -2v \end{vmatrix} = -4(u^2 + v^2), \quad \frac{\partial(z,x)}{\partial(u,v)} = \begin{vmatrix} 2u & 2v \\ 2v & 2u \end{vmatrix} = 4(u^2 - v^2),$$

$$\frac{\partial(y,z)}{\partial(u,v)} = \begin{vmatrix} 2u & -2v \\ 2u & 2v \end{vmatrix} = 8uv.$$

EXAMPLE

Find $\iint_S (x^2+y^2)\,dS$, where S is the parametric curve $x=2uv$, $y=u^2-v^2$, $z=u^2+v^2$, $u^2+v^2\le 1$.

Solution:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2v & 2u \\ 2u & -2v \end{vmatrix} = -4(u^2+v^2), \quad \frac{\partial(z,x)}{\partial(u,v)} = \begin{vmatrix} 2u & 2v \\ 2v & 2u \end{vmatrix} = 4(u^2-v^2),$$

$$\frac{\partial(y,z)}{\partial(u,v)} = \begin{vmatrix} 2u & -2v \\ 2u & 2v \end{vmatrix} = 8uv.$$

$$\boxed{dS = 4\sqrt{(u^2+v^2)^2+(u^2-v^2)^2+4u^2v^2}\,du\,dv = 4\sqrt{2}(u^2+v^2)\,du\,dv}$$

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$$\boxed{dS=4\sqrt{(u^2+v^2)^2+(u^2-v^2)^2+4u^2v^2}\,du\,dv=4\sqrt{2}(u^2+v^2)\,du\,dv}$$

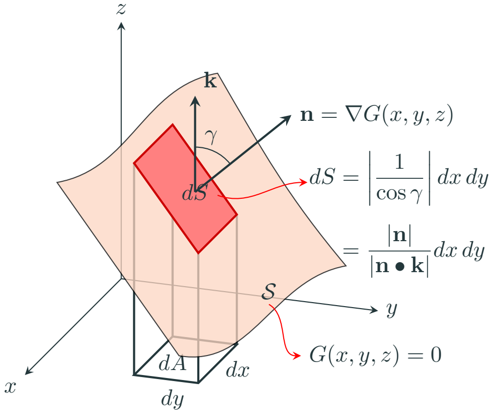
Now $x^2+y^2=4u^2v^2+(u^2-v^2)^2=(u^2+v^2)^2$, and so:

$$\iint_S (x^2+y^2)\,dS=\iint_{u^2+v^2\le 1}(u^2+v^2)^2\,4\sqrt{2}(u^2+v^2)\,du\,dv$$

$$=4\sqrt{2}\int_0^{2\pi}d\theta\int_0^1 r^7\,dr=\boxed{\sqrt{2}\pi}$$

Surfaces and Surface Integrals

Evaluating Surface Integrals



$$\mathbf{n} = \nabla G(x,y,z)$$

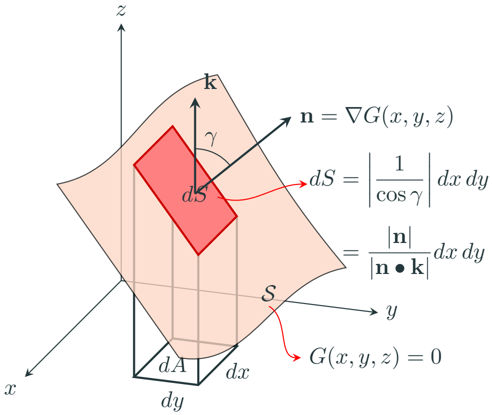
$$dS = \left|\frac{1}{\cos\gamma}\right| dx\,dy$$

$$= \frac{|\mathbf{n}|}{|\mathbf{n} \bullet \mathbf{k}|} dx\,dy$$

$$G(x,y,z) = 0$$

Surfaces and Surface Integrals

Evaluating Surface Integrals



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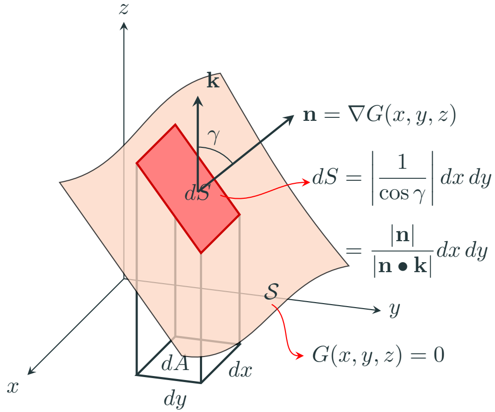
Surfaces and Surface Integrals

Evaluating Surface Integrals



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$$dS = \left|\frac{\nabla G(x, y, z)}{G_3(x, y, z)}\right| dx\,dy$$

$$\iint_{\mathcal{S}} f(x, y, z)\,dS = \iint_{D} f(x, y, g(x, y)) \left|\frac{\nabla G(x, y, z)}{G_3(x, y, z)}\right| dx\,dy.$$

the one-to-one projection of $\mathcal{S}$ onto the xy-plane

Surfaces and Surface Integrals

Example — Hyperbolic Bowl

EXAMPLE

Find the moment about $z = 0$:

$$M_{z=0} = \iint_S z \, dS,$$

where S is the hyperbolic bowl $z^2 = 1 + x^2 + y^2$, between planes $z = 1$ and $z = \sqrt{5}$.

Surfaces and Surface Integrals

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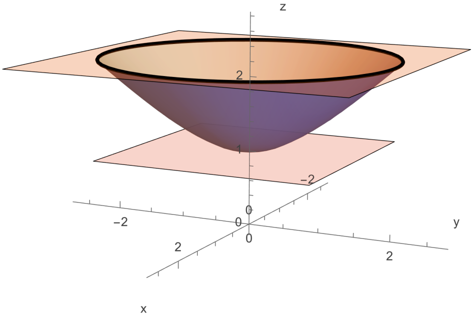

Surfaces and Surface Integrals

Example — Hyperbolic Bowl (cont.)

Solution: S is given by $G(x, y, z) = 0$, where $G(x, y, z) = x^2 + y^2 - z^2 + 1$. It lies above the disk $x^2 + y^2 \leq 4$ in the xy-plane. We have $\nabla G = 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k}$, and $G_3 = -2z$.

Surfaces and Surface Integrals

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$$z\,dS = z\frac{\sqrt{4x^2 + 4y^2 + 4z^2}}{2z}dx\,dy = \sqrt{1 + 2(x^2 + y^2)}dx\,dy,$$

Surfaces and Surface Integrals

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$$z\,dS = z\frac{\sqrt{4x^2 + 4y^2 + 4z^2}}{2z}dx\,dy = \sqrt{1 + 2(x^2 + y^2)}dx\,dy,$$

and the required moment is

$$\begin{aligned} M_{z=0} &= \iint_{x^2+y^2\leq 4} \sqrt{1 + 2(x^2 + y^2)}\,dx\,dy \\ &= \int_0^{2\pi} d\theta \int_0^2 \sqrt{1 + 2r^2}\,r\,dr = \frac{\pi}{3}(1 + 2r^2)^{3/2}\Big|_0^2 = \frac{26\pi}{3}. \end{aligned}$$

Oriented Surfaces and Flux Integrals

Oriented Surfaces and Flux Integrals

Oriented Surfaces

Definition

A smooth surface $\mathcal{S}$ in 3-space is said to be **orientable** if there exists a unit vector field $\hat{\mathbf{N}}(P)$ defined on $\mathcal{S}$ that varies *continuously* as P ranges over $\mathcal{S}$ and that is everywhere normal to $\mathcal{S}$.

Oriented Surfaces and Flux Integrals

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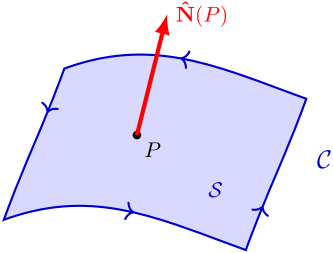
- Any such field $\hat{\mathbf{N}}(P)$ determines an **orientation** of $\mathcal{S}$.
- The surface must have **two sides** since $\hat{\mathbf{N}}(P)$ can have only one value at each point.
- The side out of which $\hat{\mathbf{N}}$ points is the **positive side**; the other is the **negative side**.
- $(\mathcal{S}, \hat{\mathbf{N}})$ is called an **oriented surface**.



Oriented Surfaces and Flux Integrals

Boundary Orientation

A smooth or piecewise smooth surface may be **closed** (that is, it may have no boundary), or it may have one or more boundary curves. The unit normal vector field $\widehat{\mathbf{N}}(P)$ need not be defined at points of the boundary curves.

Oriented Surfaces and Flux Integrals

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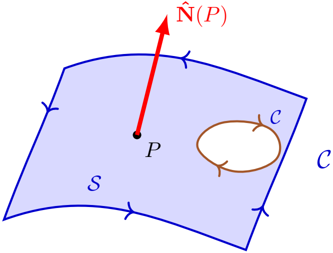
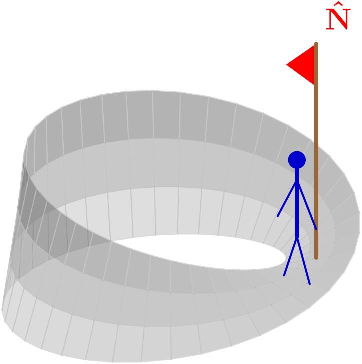
Boundary Orientation

An oriented surface $\mathcal{S}$ **induces an orientation** on any of its boundary curves $\mathcal{C}$: if we stand on the **positive side** of $\mathcal{S}$ and walk around $\mathcal{C}$ in the direction of its orientation, then the surface $\mathcal{S}$ will be on our **left**.



Oriented Surfaces

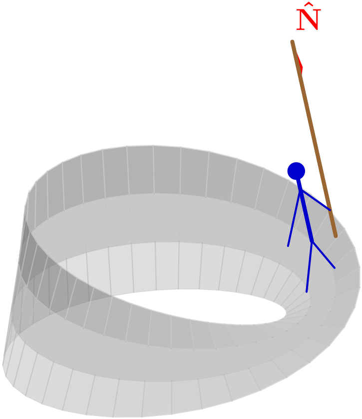
Animation: Walking on the Möbius Band



Walking along the surface with a flag pointing in the normal direction...

Oriented Surfaces

Animation: Walking on the Möbius Band



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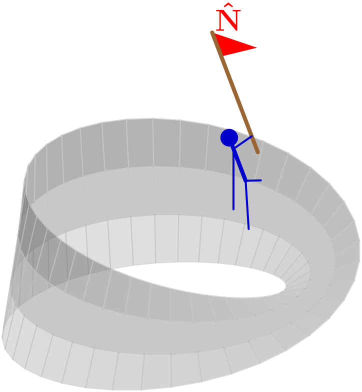

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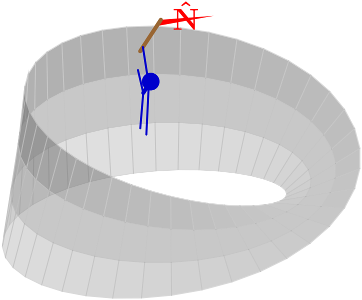

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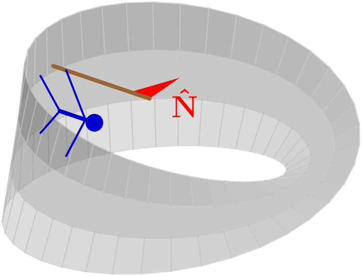

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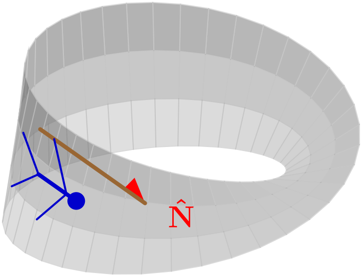

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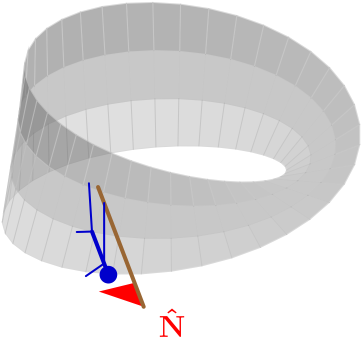

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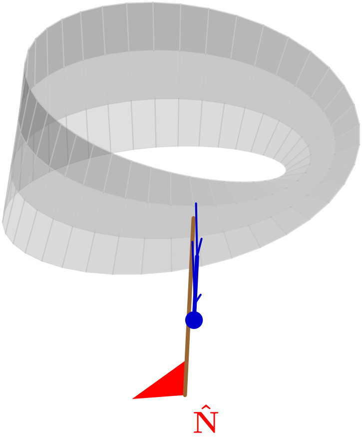

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Oriented Surfaces

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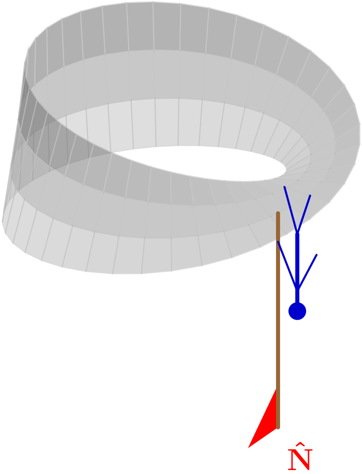
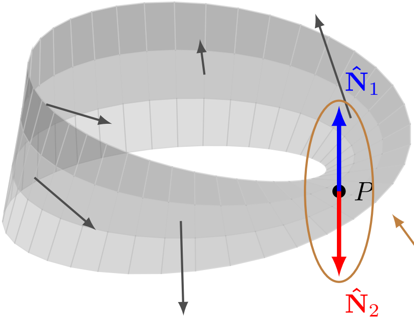


After one full trip, the walker and the flag are completely UPSIDE DOWN!

Oriented Surfaces

Example: The Möbius Band



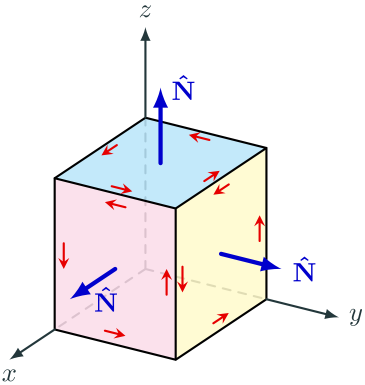
The unit normal vector does not change continuously at P.

The Möbius band is not orientable; it has only one "side"

Oriented Surfaces and Flux Integrals

Piecewise Smooth Surfaces

A *piecewise smooth* surface is **orientable** if, whenever two smooth component surfaces join along a common boundary curve $\mathcal{C}$, they induce **opposite orientations** along $\mathcal{C}$. This forces the normals to be on the same side of adjacent components.



The surface of the cube is orientable; adjacent faces induce opposite orientations on their common edge.

Oriented Surfaces and Flux Integrals

The Flux of a Vector Field Across a Surface

- Suppose that 3-space is filled with an incompressible fluid that flows with velocity field $\mathbf{v}$.

Oriented Surfaces and Flux Integrals

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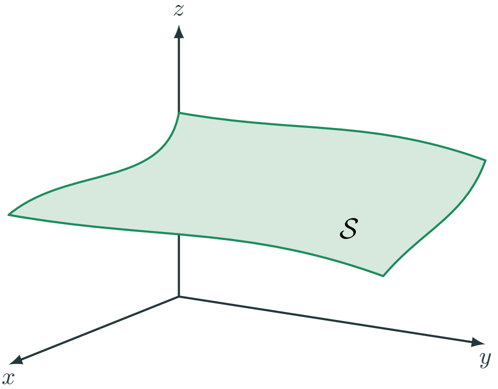
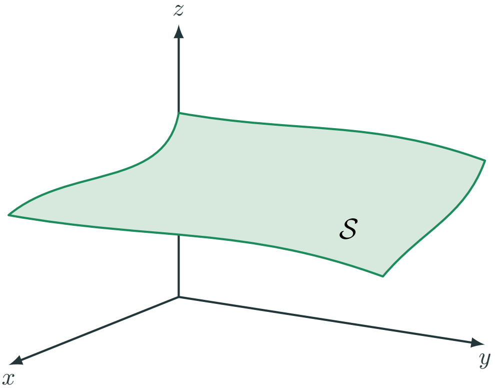
- Suppose that 3-space is filled with an incompressible fluid that flows with velocity field $\mathbf{v}$.
- Let $\mathcal{S}$ be an imaginary, smooth, oriented surface in 3-space.



Oriented Surfaces and Flux Integrals

The Flux of a Vector Field Across a Surface

- Suppose that 3-space is filled with an incompressible fluid that flows with velocity field $\mathbf{v}$.
- Let $\mathcal{S}$ be an imaginary, smooth, oriented surface in 3-space.
- We want to calculate the rate at which fluid flows across $\mathcal{S}$.



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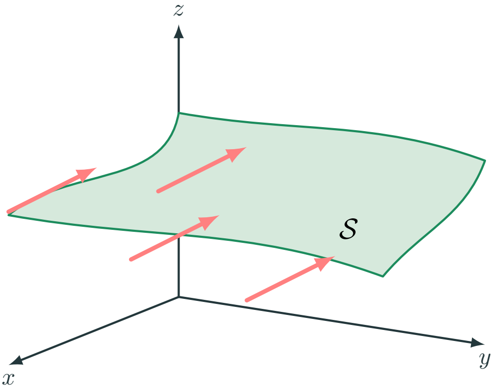

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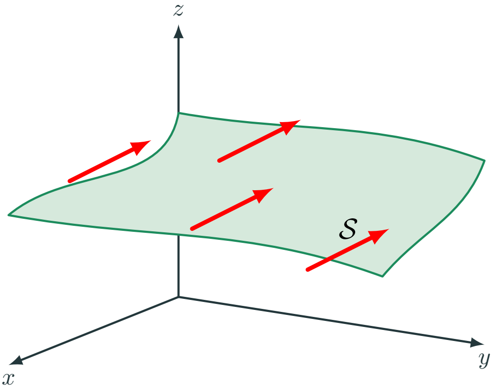

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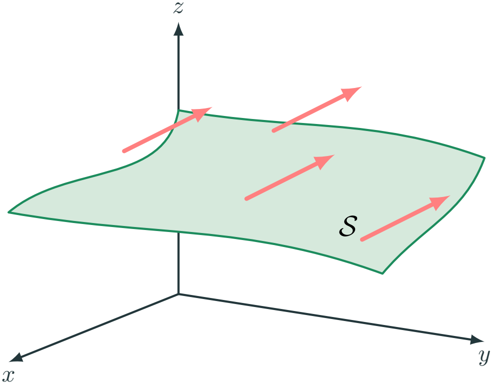

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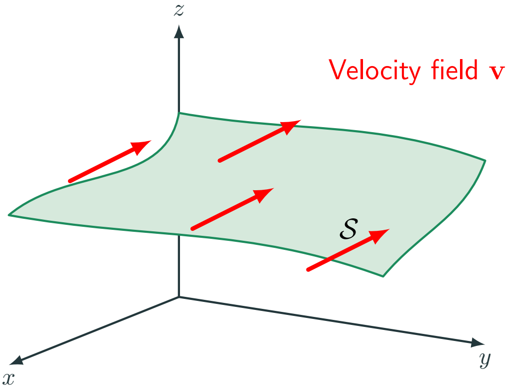
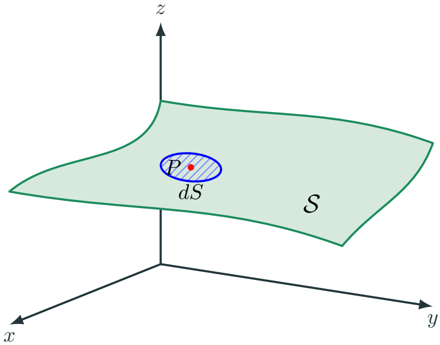


Oriented Surfaces and Flux Integrals

Calculating the Flow Rate Across an Area Element

- Let dS be a small area element at a point P on the surface $\mathcal{S}$.



Oriented Surfaces and Flux Integrals

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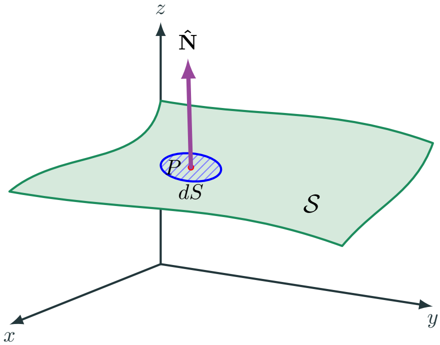

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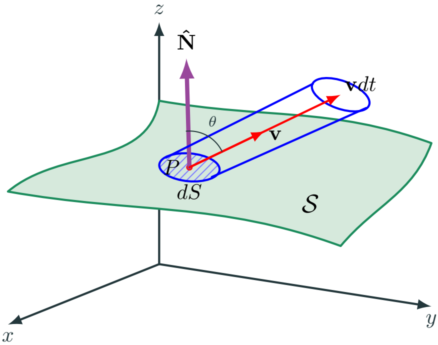
- If the fluid velocity at P is $\mathbf{v}(P)$ and the angle between $\mathbf{v}(P)$ and $\widehat{\mathbf{N}}(P)$ is θ, then in a short time interval dt, the fluid crossing that element between time t and time $t + dt$ occupies approximately a cylinder of base dS and height $|\mathbf{v}(P)| \cos\theta \, dt$.



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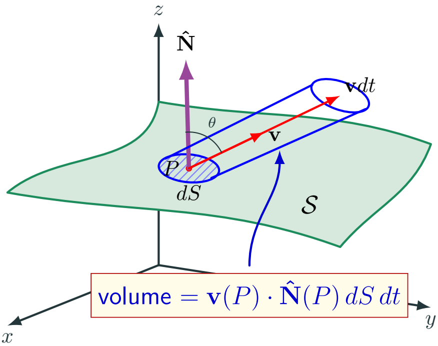

volume $= \mathbf{v}(P) \cdot \hat{\mathbf{N}}(P)\, dS\, dt$

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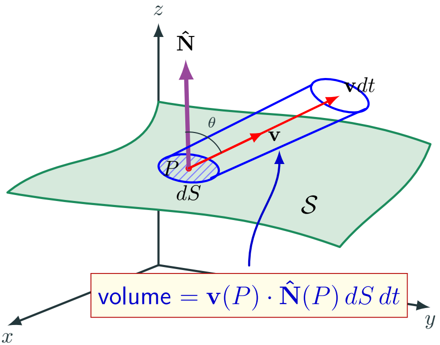


volume $= \mathbf{v}(P) \cdot \hat{\mathbf{N}}(P)\, dS\, dt$

So the **rate** at which fluid crosses the small patch is $\mathbf{v}(P) \cdot \widehat{\mathbf{N}}(P)\, dS$.

Oriented Surfaces and Flux Integrals

The Flux of a Vector Field Across a Surface

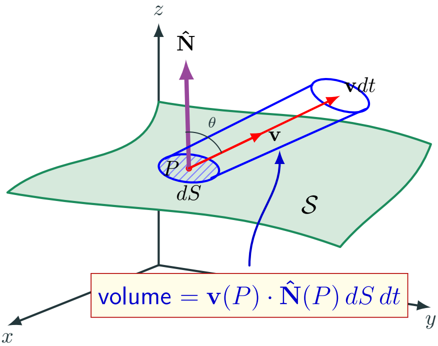
The rate at which fluid crosses dS is $\mathbf{v}(P)\cdot\widehat{\mathbf{N}}(P)\,dS$, and the total rate at which it crosses $\mathcal{S}$ is the sum of these contributions over all area elements dS of $\mathcal{S}$:

$$\iint_{\mathcal{S}} \mathbf{v}\cdot\widehat{\mathbf{N}}\,dS.$$

or

$$\iint_{\mathcal{S}} \mathbf{v}\cdot d\mathbf{S},$$

where $d\mathbf{S}$ to represent the vector surface area elenement $\widehat{\mathbf{N}}\,dS$.



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Oriented Surfaces and Flux Integrals

The Flux of a Vector Field Across a Surface

Definition. *Flux of a vector field across an oriented surface*

Given any continuous vector field $\mathbf{F}$, the **flux** of $\mathbf{F}$ across the orientable surface $\mathcal{S}$ is the integral of the normal component of $\mathbf{F}$ over $\mathcal{S}$:

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \widehat{\mathbf{N}}\, dS \qquad \text{or} \qquad \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S},$$

where

$$d\mathbf{S} = \widehat{\mathbf{N}}\, dS.$$

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where

$$d\mathbf{S} = \widehat{\mathbf{N}}\, dS.$$

When the surface is closed, the flux integral is often denoted by

$$\oiint_{\mathcal{S}} \mathbf{F} \cdot \widehat{\mathbf{N}}\, dS \qquad \text{or} \qquad \oiint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}.$$

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Definition. *Flux of a vector field across an oriented surface*

Given any continuous vector field $\mathbf{F}$, the **flux** of $\mathbf{F}$ across the orientable surface $\mathcal{S}$ is the integral of the normal component of $\mathbf{F}$ over $\mathcal{S}$:

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \widehat{\mathbf{N}}\, dS \qquad \text{or} \qquad \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S},$$

where

$$d\mathbf{S} = \widehat{\mathbf{N}}\, dS.$$

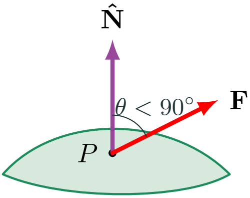
When the surface is closed, the flux integral is often denoted by

$$\oiint_{\mathcal{S}} \mathbf{F} \cdot \widehat{\mathbf{N}}\, dS \qquad \text{or} \qquad \oiint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}.$$

- If $\widehat{\mathrm{N}}$ is the **unit exterior normal**, we speak of the flux of $\mathbf{F}$ **out of** $\mathcal{S}$.
- If $\widehat{\mathrm{N}}$ is the **unit interior normal**, we speak of the flux of $\mathbf{F}$ **into** $\mathcal{S}$.

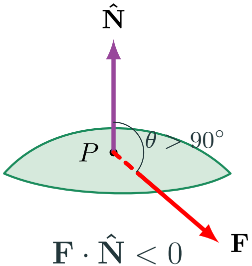
Oriented Surfaces and Flux Integrals

Interpreting the Sign of Flux



$$\mathbf{F} \cdot \hat{\mathbf{N}} > 0$$

Positive Flux



$$\mathbf{F} \cdot \hat{\mathbf{N}} < 0$$

Negative Flux

Interpretation

Flux measures how much of the vector field passes through the surface in the direction of the chosen normal. Because $\mathbf{F} \cdot \hat{\mathbf{N}} = |\mathbf{F}||\hat{\mathbf{N}}| \cos\theta$, the flux is positive when the angle θ is acute, and negative when θ is obtuse. Changing the orientation changes the sign of the flux.

Oriented Surfaces and Flux Integrals

The Flux of a Vector Field Across a Surface

EXAMPLE

Find the flux of the vector field $\mathbf{F} = \dfrac{m\mathbf{r}}{|\mathbf{r}|^3}$ out of a sphere $\mathcal{S}$ of radius a centred at the origin. (Here $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.)

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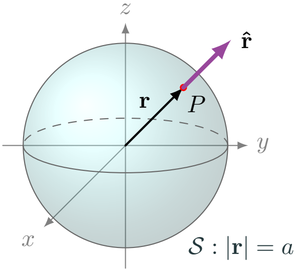
Solution: We use spherical coordinates. At any point $\mathbf{r}$ on the sphere, with spherical coordinates $[a, \phi, \theta]$, the unit outward normal is $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$.



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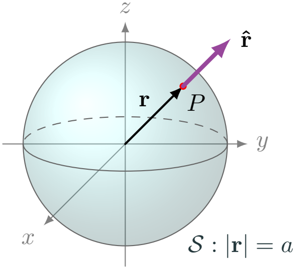

Since the vector field is $\mathbf{F} = m\hat{\mathbf{r}}/a^2$ on the sphere, and since an area element is

$$dS = a^2 \sin\phi \, d\phi \, d\theta,$$

the flux of $\mathbf{F}$ out of the sphere is

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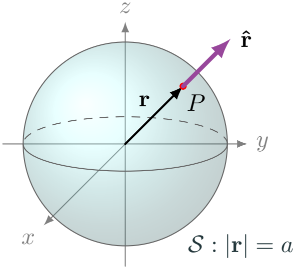

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$$dS = a^2 \sin\phi \, d\phi \, d\theta,$$

the flux of $\mathbf{F}$ out of the sphere is

$$\iint_{\mathcal{S}} \left(\frac{m}{a^2}\hat{\mathbf{r}}\right) \cdot \hat{\mathbf{r}} \, a^2 \sin\phi \, d\phi \, d\theta = m \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi \, d\phi = \boxed{4\pi m}$$

Oriented Surfaces and Flux Integrals

The Flux of a Vector Field Across a Surface

EXAMPLE

Calculate the total flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ outward through the surface of the solid cylinder $x^2 + y^2 \leq a^2$, $-h \leq z \leq h$.

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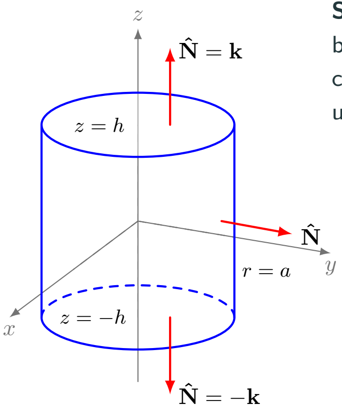

Solution. The cylinder surface consists of top and bottom disks and the cylindrical side wall. We calculate the flux of $\mathbf{F}$ out of each. Naturally, we use cylindrical coordinates.

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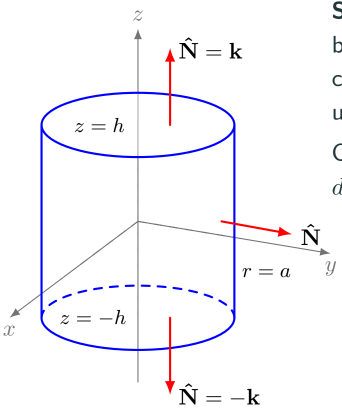

Solution. The cylinder surface consists of top and bottom disks and the cylindrical side wall. We calculate the flux of $\mathbf{F}$ out of each. Naturally, we use cylindrical coordinates.

On the **top disk** we have $z = h$, $\hat{\mathbf{N}} = \mathbf{k}$, and $dS = r\,dr\,d\theta$. Therefore, $\mathbf{F} \cdot \hat{\mathbf{N}}\,dS = hr\,dr\,d\theta$ and

$$\iint_{\text{top}} \mathbf{F} \cdot \hat{\mathbf{N}}\,dS = h \int_0^{2\pi} d\theta \int_0^a r\,dr = \pi a^2 h.$$

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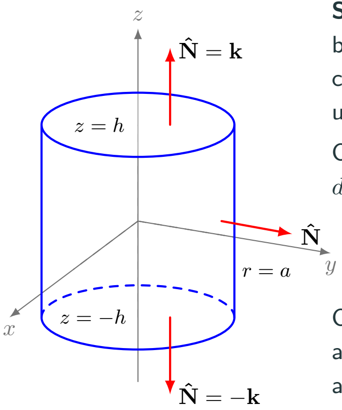

Solution. The cylinder surface consists of top and bottom disks and the cylindrical side wall. We calculate the flux of $\mathbf{F}$ out of each. Naturally, we use cylindrical coordinates.

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On the **bottom disk** we have $z = -h$, $\hat{\mathbf{N}} = -\mathbf{k}$, and $dS = r\,dr\,d\theta$. Therefore, $\mathbf{F} \cdot \hat{\mathbf{N}}\,dS = hr\,dr\,d\theta$ and

$$\iint_{\text{bottom}} \mathbf{F} \cdot \hat{\mathbf{N}}\,dS = \iint_{\text{top}} \mathbf{F} \cdot \hat{\mathbf{N}}\,dS = \pi a^2 h.$$

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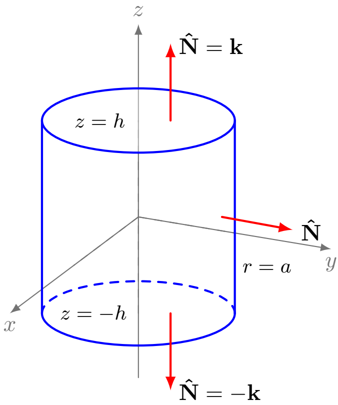

On the **cylindrical wall**

$\mathbf{F} = a\cos\theta\,\mathbf{i} + a\sin\theta\,\mathbf{j} + z\mathbf{k}$, $\hat{\mathbf{N}} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$, and $dS = a\,d\theta\,dz$.

Thus, $\mathbf{F} \cdot \hat{\mathbf{N}}\,dS = a^2\,d\theta\,dz$ and

$$\iint_{\text{cylwall}} \mathbf{F} \cdot \hat{\mathbf{N}}\,dS = a^2 \int_0^{2\pi} d\theta \int_{-h}^{h} dz = 4\pi a^2 h.$$

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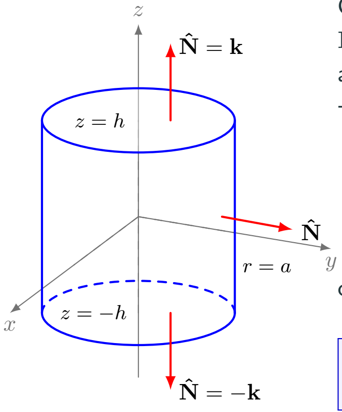
Calculate the total flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ outward through the surface of the solid cylinder $x^2 + y^2 \le a^2$, $-h \le z \le h$.



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Thus, $\mathbf{F}\cdot\hat{\mathbf{N}}\,dS = a^2\,d\theta\,dz$ and

$$\iint_{\text{cylwall}} \mathbf{F}\cdot\hat{\mathbf{N}}\,dS = a^2\int_0^{2\pi} d\theta \int_{-h}^{h} dz = 4\pi a^2 h.$$

The total flux of $\mathbf{F}$ out of the surface $\mathcal{S}$ of the cylinder is the sum of these three contributions:

$$\boxed{\iint_{\mathcal{S}} \mathbf{F}\cdot\hat{\mathbf{N}}\,dS = \pi a^2 h + \pi a^2 h + 4\pi a^2 h = \mathbf{6\pi a^2 h}.}$$