MAT124 MATHEMATICS II

Line Integrals

Outline

Evaluating Line Integrals

Line Integrals of Vector Fields

Evaluating Line Integrals

EXAMPLE

Find the centroid of the circular helix C given by

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The helix length is $L = \int_C ds = \int_0^{2\pi} \sqrt{a^2+b^2}\,dt = 2\pi\sqrt{a^2+b^2}$. Hence

$$\bar{z} = \frac{M_{z=0}}{L} = \pi b.$$

Solution: The other two moments are

$$M_{x=0} = \int_C x\, ds = a\sqrt{a^2+b^2}\int_0^{2\pi} \cos t\, dt = 0,$$

$$M_{y=0} = \int_C y\, ds = a\sqrt{a^2+b^2}\int_0^{2\pi} \sin t\, dt = 0.$$

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The centroid of the helix is

$$(\bar{x}, \bar{y}, \bar{z}) = (0, 0, \pi b).$$

EXAMPLE

Find the mass of a wire lying along the first-octant part C of the intersection

$$z = 2 - x^2 - 2y^2 \qquad \text{and} \qquad z = x^2,$$

between $(0, 1, 0)$ and $(1, 0, 1)$, if the density is $\delta(x, y, z) = xy$.

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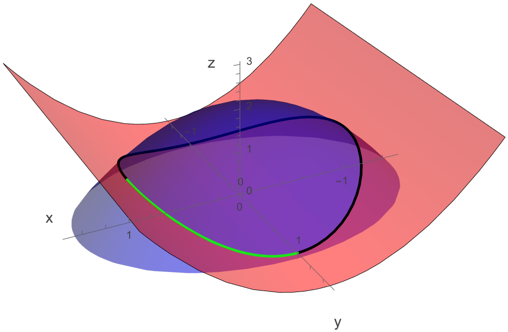

EXAMPLE (continued)
Find the mass of the wire on C with density $\delta(x, y, z) = xy$.

Solution: Use the cylinder equation $z = x^2$ and let

$$x = t, \qquad z = t^2, \qquad 0 \le t \le 1.$$

Substitute into $z = 2 - x^2 - 2y^2$:

$$t^2 = 2 - t^2 - 2y^2 \quad \Longrightarrow \quad y^2 = 1 - t^2.$$

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$$y = \sqrt{1 - t^2}.$$

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So a parametrization is

$$\mathbf{r}(t) = t\,\mathbf{i} + \sqrt{1 - t^2}\,\mathbf{j} + t^2\,\mathbf{k}, \qquad 0 \leq t \leq 1.$$

Solution: Compute derivatives:

$$\frac{dx}{dt} = 1, \qquad \frac{dy}{dt} = -\frac{t}{\sqrt{1-t^2}}, \qquad \frac{dz}{dt} = 2t.$$

Hence

$$ds = \left|\frac{d\mathbf{r}}{dt}\right| dt = \sqrt{1 + \frac{t^2}{1-t^2} + 4t^2}\, dt = \sqrt{\frac{1+4t^2-4t^4}{1-t^2}}\, dt.$$

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Also,

$$\delta(x, y, z) = xy = t\sqrt{1-t^2}.$$

Therefore the mass integral becomes

$$m = \int_C \delta\, ds = \int_0^1 t\sqrt{1+4t^2-4t^4}\, dt.$$

Line Integrals

Evaluating Line Integrals — Wire Mass (cont.)

Solution: Evaluate: $m = \int_0^1 t\sqrt{1 + 4t^2 - 4t^4}\, dt$.

Let $u = t^2, \quad du = 2t\, dt$:

$$m = \frac{1}{2} \int_0^1 \sqrt{1 + 4u - 4u^2}\, du.$$

Solution: Evaluate: $m = \int_0^1 t\sqrt{1+4t^2-4t^4}\,dt$.

Let $u = t^2, \quad du = 2t\,dt$:

$$m = \frac{1}{2}\int_0^1 \sqrt{1+4u-4u^2}\,du.$$

Complete the square: $1+4u-4u^2 = 2-(2u-1)^2$.

Let $v = 2u-1, \quad dv = 2\,du$:

$$m = \frac{1}{4}\int_{-1}^1 \sqrt{2-v^2}\,dv = \frac{1}{2}\int_0^1 \sqrt{2-v^2}\,dv.$$

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The mass of the wire is $m = \dfrac{\pi+2}{8}$.

Line Integrals of Vector Fields

In elementary physics, the work done by a **constant force** of magnitude F in moving an object a distance d is

$$W = Fd$$

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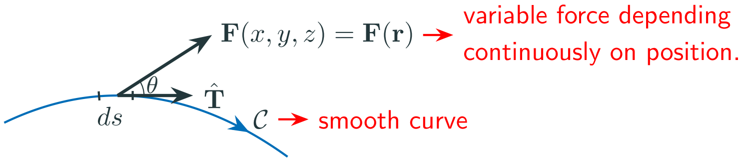
If the object moves in a direction *different* from that of the force (because of some other forces acting on it), then the work done by the particular force is the product of the distance moved and the **component of the force in the direction of motion**.

Line Integrals of Vector Fields

Work Done by a Variable Force

Let $\mathbf{F}(x, y, z) = \mathbf{F}(\mathbf{r})$ be a **variable force** depending continuously on position, and let C be a smooth curve.



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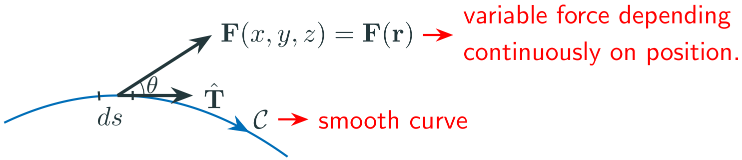

The work done by $\mathbf{F}(\mathbf{r})$ in moving an object along C is the integral of work elements dW:

$$dW = \underbrace{\mathbf{F}(\mathbf{r}) \cdot \hat{\mathbf{T}}}_{|F(\mathbf{r})|\cos\theta = \text{the tangential component of } F(\mathbf{r})} ds = \mathbf{F} \cdot d\mathbf{r} \qquad (\text{since } \hat{\mathbf{T}} = d\mathbf{r}/ds)$$

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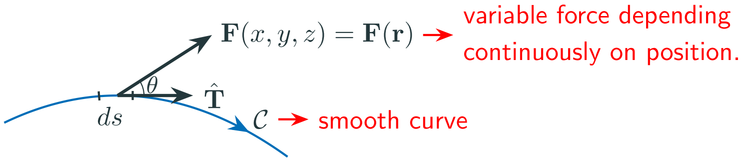

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Thus, the total work done by $\mathbf{F}$ in moving the object along C is

$$W = \int_C \mathbf{F} \cdot \hat{\mathbf{T}}\, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C F_1\, dx + F_2\, dy + F_3\, dz.$$

In general, if $\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$ is a continuous vector field, and C is an oriented smooth curve, then the **line integral of the tangential component of $\mathbf{F}$ along** C is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \hat{\mathbf{T}}\,ds = \int_C F_1(x, y, z)\,dx + F_2(x, y, z)\,dy + F_3(x, y, z)\,dz.$$

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Such a line integral is sometimes called, somewhat improperly, the *line integral of* $\mathbf{F}$ *along* C. (It is not the line integral of $\mathbf{F}$, which should have a vector value, *but rather the line integral of the tangential component of* $\mathbf{F}$, which has a scalar value.)

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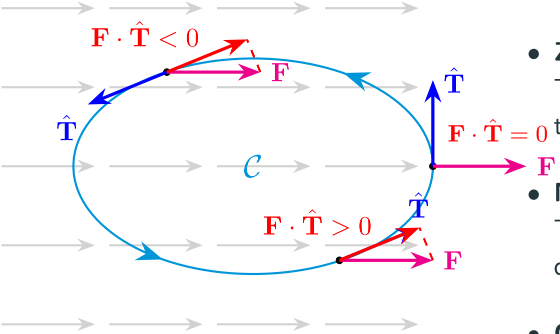
The line integral of a vector field depends on the **direction of the orientation** of the curve along which the integral is calculated.

If C is a **closed curve**, the line integral of the tangential component of $\mathbf{F}$ around C is also called the **circulation** of $\mathbf{F}$ around C.

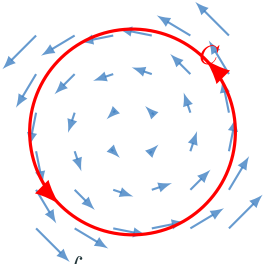
The fact that the curve is closed is often indicated by a small circle drawn on the integral sign:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} \quad \text{denotes the circulation of } \mathbf{F} \text{ around the closed curve } C.$$



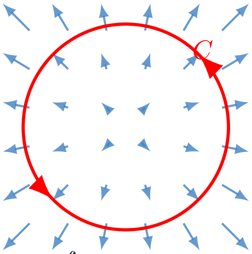
- **Positive Contribution:** The field ($\mathbf{F}$) supports the direction of motion ($\hat{\mathbf{T}}$).
- **Zero Contribution:** The field is orthogonal to the path (does no work).
- **Negative Contribution:** The field opposes the direction of motion.
- **Circulation:** The net sum of all these tangential components along the closed path!

Positive Circulation



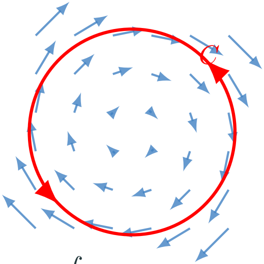
$$\oint_C \mathbf{F} \cdot d\mathbf{r} > 0$$

Zero Circulation



$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

Negative Circulation



$$\oint_C \mathbf{F} \cdot d\mathbf{r} < 0$$

For a smooth arc $\mathbf{r} = \mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}, \quad a \le t \le b$, we have

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\,dt \\ &= \int_a^b \Big[F_1\big(x(t), y(t), z(t)\big)\,\tfrac{dx}{dt} + F_2\big(x(t), y(t), z(t)\big)\,\tfrac{dy}{dt} \\ &\quad + F_3\big(x(t), y(t), z(t)\big)\,\tfrac{dz}{dt}\Big]\,dt.\end{aligned}$$

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Remarks

- The above line integral is **independent of the choice of any parametrization** used for the curve with a fixed orientation.
- A line integral over a **piecewise smooth** curve is the sum of the line integrals over the smooth arcs constituting the curve.

EXAMPLE

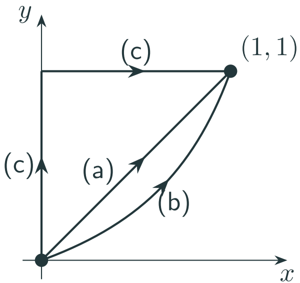
Let $\mathbf{F}(x, y) = y^2\,\mathbf{i} + 2xy\,\mathbf{j}$. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ from $(0, 0)$ to $(1, 1)$ along

(a) the straight line $y = x$,

(b) the curve $y = x^2$, and

(c) the piecewise smooth path consisting of the straight line segments from $(0, 0)$ to $(0, 1)$ and from $(0, 1)$ to $(1, 1)$.

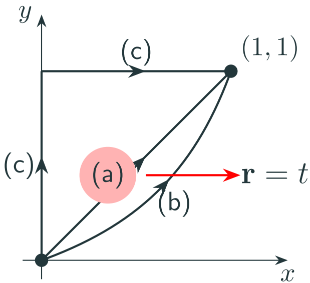
Solution:



Three paths from

$(0, 0)$ to $(1, 1)$

Solution:



$$\mathbf{r} = t\mathbf{i} + t\mathbf{j}, \quad 0 \le t \le 1$$

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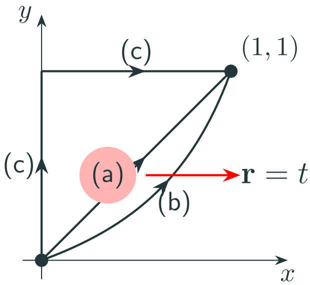

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$$\mathbf{r} = t\mathbf{i} + t\mathbf{j}, \quad 0 \le t \le 1 \longrightarrow d\mathbf{r} = dt\mathbf{i} + dt\mathbf{j}$$

$$\mathbf{F} \cdot d\mathbf{r} = (t^2\mathbf{i} + 2t^2\mathbf{j}) \cdot (\mathbf{i} + \mathbf{j})dt = 3t^2\, dt.$$

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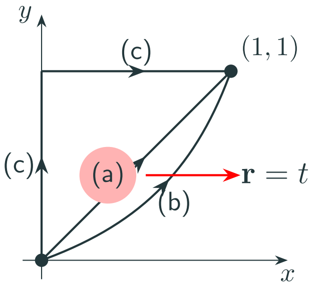

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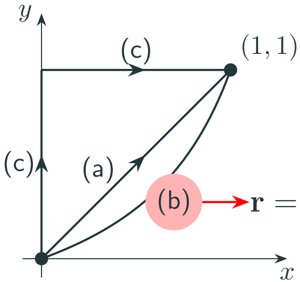
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 3t^2\, dt = t^3\Big|_0^1 = 1.$$

Line Integrals of Vector Fields

Example — Three Paths: (b)

Solution: (b)



$$\mathbf{r} = t\mathbf{i} + t^2\mathbf{j}, \quad 0 \le t \le 1$$

Three paths from

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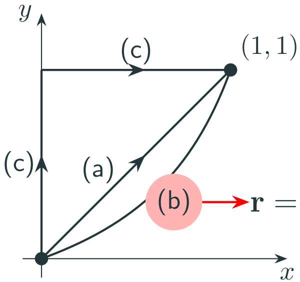

$$\mathbf{r} = t\mathbf{i} + t^2\mathbf{j}, \quad 0 \le t \le 1 \longrightarrow d\mathbf{r} = dt\mathbf{i} + 2t\,dt\mathbf{j}$$

$$\mathbf{F} \cdot d\mathbf{r} = (t^4\mathbf{i} + 2t^3\mathbf{j}) \cdot (\mathbf{i} + 2t\mathbf{j})dt = 5t^4\,dt.$$

Three paths from $(0, 0)$ to $(1, 1)$

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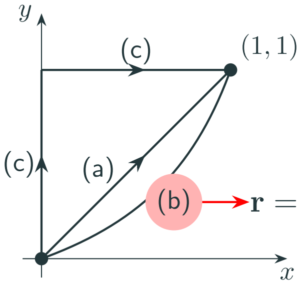

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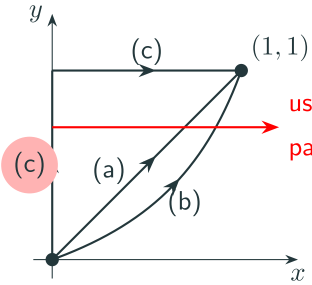
$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 5t^4\,dt = t^5\Big|_0^1 = 1.$$

Solution:

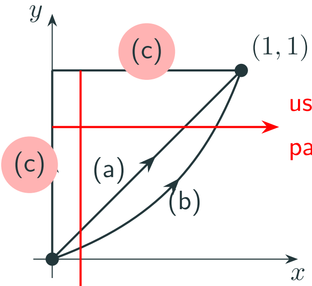


Three paths from
$(0,0)$ to $(1,1)$

use y as the
parameter (where $x = 0$) $\rightarrow d\mathbf{r} = dy\mathbf{j}$

$$\mathbf{F} \cdot d\mathbf{r} = 2xy\,dy$$

Solution:



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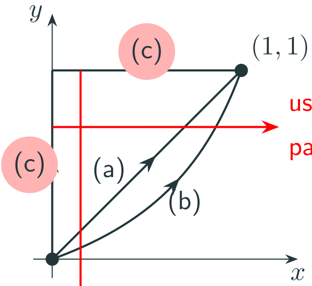

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$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C y^2\,dx + 2xy\,dy$$

$$= \int_0^1 (0)\,dy + \int_0^1 (1)\,dx = 1.$$

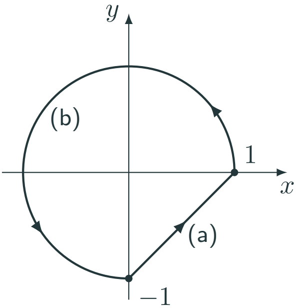
EXAMPLE

Let $\mathbf{F} = y\,\mathbf{i} - x\,\mathbf{j}$. Find $\int_C \mathbf{F} \cdot d\mathbf{r}$ from $(1, 0)$ to $(0, -1)$ along

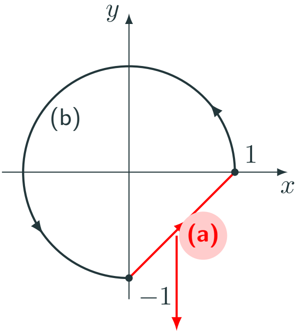
(a) the straight line segment joining these points, and

(b) three-quarters of the circle of unit radius centred at the origin and traversed counterclockwise.

Solution:



Solution:



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$$0 \le t \le 1.$$

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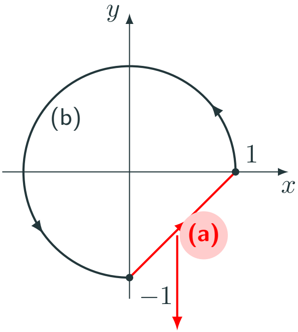

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$$d\mathbf{r} = -dt\mathbf{i} - dt\mathbf{j}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \big((-t)(-dt) - (1-t)(-dt)\big)$$

$$= \int_0^1 dt = 1.$$

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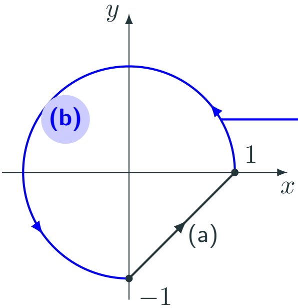

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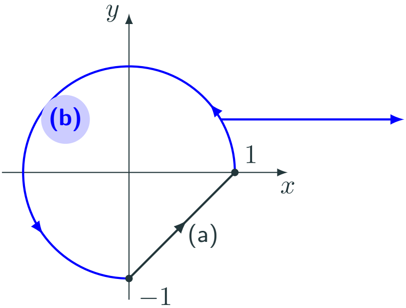

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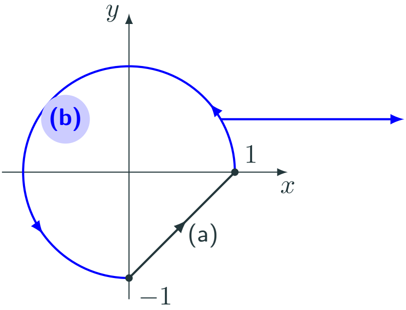

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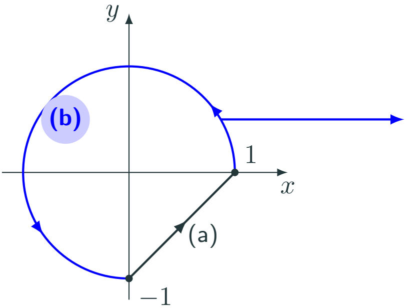

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Note

Unlike the previous example, different paths give **different** answers.

Domain

A set D in the plane or 3-space is called a **domain** if $D = S \cup B$, where S is an open set and B is a (possibly empty) set of boundary points of S.

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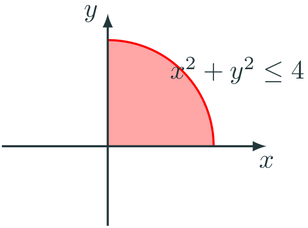
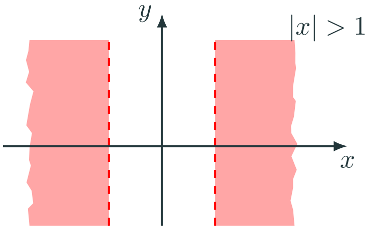


connected



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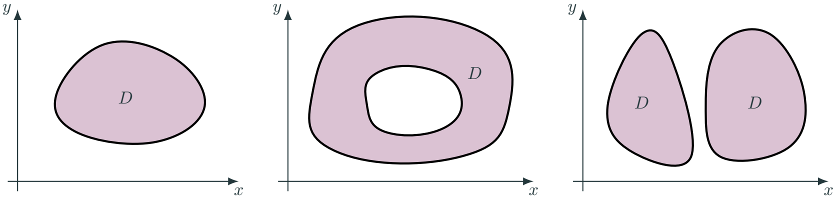

A simply connected domain

A connected domain that is not simply connected

A domain that is not connected

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- In **3-space**, a simply connected domain *can* have holes.
  - The exterior of a ball is simply connected.
  - But the set of all points in $\mathbb{R}^3$ satisfying $x^2 + y^2 > 0$ is **not** simply connected.

Theorem

Let D be an open, connected domain, and let $\mathbf{F}$ be a smooth vector field defined on D. Then the following three statements are **equivalent** (if any one is true, so are the other two):

(a) $\mathbf{F}$ is **conservative** in D.

(b) $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every piecewise smooth, closed curve C in D.

(c) Given any two points P_0 and P_1 in D, $\int_C \mathbf{F} \cdot d\mathbf{r}$ has the same value for all piecewise smooth curves in D starting at P_0 and ending at P_1.

Remark

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If $\mathbf{F} = \nabla\phi$ and C goes from P_0 to P_1, then

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Because

$$\begin{aligned}\mathbf{F}\cdot d\mathbf{r} &= \left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right)\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})\\ &= \frac{\partial\phi}{\partial x}\,dx + \frac{\partial\phi}{\partial y}\,dy + \frac{\partial\phi}{\partial z}\,dz = d\phi,\end{aligned}$$

$$\Rightarrow \int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \frac{d}{dt}\left[\phi(\mathbf{r}(t))\right]dt = \phi\big(\mathbf{r}(b)\big) - \phi\big(\mathbf{r}(a)\big).$$

EXAMPLE

For what values of the constants A and B is the vector field

$$\mathbf{F} = Ax\sin(\pi y)\,\mathbf{i} + \left(x^2\cos(\pi y) + By\,e^{-z}\right)\mathbf{j} + y^2 e^{-z}\,\mathbf{k}$$

conservative? For this choice of A and B, evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is

(a) the curve $\mathbf{r} = \cos t\,\mathbf{i} + \sin 2t\,\mathbf{j} + \sin^2 t\,\mathbf{k}, \quad 0 \le t \le 2\pi$, and

(b) the curve of intersection of the paraboloid $z = x^2 + 4y^2$ and the plane $z = 3x - 2y$ from $(0,0,0)$ to $(1, \frac{1}{2}, 2)$.

Solution: $\mathbf{F}$ cannot be conservative unless

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \qquad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}.$$

Line Integrals of Vector Fields

Example — Finding A and B

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Computing each:

$$A\pi x \cos(\pi y) = 2x \cos(\pi y), \qquad 0 = 0, \qquad -By\, e^{-z} = 2y\, e^{-z}.$$

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One can verify that $\mathbf{F} = \nabla\phi$, where

$$\phi(x, y, z) = \frac{x^2}{\pi}\sin(\pi y) - y^2 e^{-z}.$$

Solution: (a) We have $\mathbf{r}(0) = \mathbf{i} = \mathbf{r}(2\pi)$, so this curve is a **closed** curve, and

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