MAT124 MATHEMATICS II

Conservative Fields & Line Integrals

Outline

Conservative Fields

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Recall that the gradient of a scalar field $\phi(x, y, z)$ is a vector field:

$$\mathbf{F}(x, y, z) = \nabla\phi(x, y, z) = \frac{\partial \phi}{\partial x}\,\mathbf{i} + \frac{\partial \phi}{\partial y}\,\mathbf{j} + \frac{\partial \phi}{\partial z}\,\mathbf{k}.$$

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What about the converse? Given a vector field $\mathbf{F}$, can we find a scalar field ϕ such that $\mathbf{F} = \nabla\phi$?

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Definition

If $\mathbf{F}(x, y, z) = \nabla\phi(x, y, z)$ in a domain D, then we say that $\mathbf{F}$ is a conservative vector field in D, and we call the function ϕ a **(scalar) potential** for $\mathbf{F}$ on D.

Similar definitions hold in the plane or in n-space.

EXAMPLE

Show that the gravitational field

$$\mathbf{F}(\mathbf{r}) = \frac{-km(\mathbf{r}-\mathbf{r}_0)}{|\mathbf{r}-\mathbf{r}_0|^3}$$

is conservative wherever it is defined (i.e., everywhere in $\mathbb{R}^3$ except at $\mathbf{r}_0$), by showing that

$$\phi(x,y,z) = \frac{km}{|\mathbf{r}-\mathbf{r}_0|} = \frac{km}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}$$

is a potential function for $\mathbf{F}$.

EXAMPLE (continued)

Show that $\phi(x, y, z) = \dfrac{km}{|\mathbf{r} - \mathbf{r}_0|}$ is a potential for $\mathbf{F}(\mathbf{r}) = \dfrac{-km(\mathbf{r} - \mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|^3}$.

Solution: Observe that

$$\frac{\partial \phi}{\partial x} = \frac{-km(x - x_0)}{\left((x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\right)^{3/2}} = \frac{-km(x - x_0)}{|\mathbf{r} - \mathbf{r}_0|^3} = F_1(x, y, z),$$

and similar formulas hold for the other partial derivatives of ϕ.

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Show that $\phi(x,y,z) = \dfrac{km}{|\mathbf{r}-\mathbf{r}_0|}$ is a potential for $\mathbf{F}(\mathbf{r}) = \dfrac{-km(\mathbf{r}-\mathbf{r}_0)}{|\mathbf{r}-\mathbf{r}_0|^3}$.

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and similar formulas hold for the other partial derivatives of ϕ.

It follows that $\nabla\phi(x,y,z) = \mathbf{F}(x,y,z)$ for $(x,y,z) \neq (x_0,y_0,z_0)$, and $\mathbf{F}$ is conservative except at $\mathbf{r}_0$.

Conservative Fields

Useful Remark

Remark. Recall the useful identity:

$$\frac{\partial}{\partial x_i}|\mathbf{F}| = \frac{\partial}{\partial x_i}\sqrt{\mathbf{F}\cdot\mathbf{F}} = \frac{2\,\mathbf{F}\cdot\dfrac{\partial \mathbf{F}}{\partial x_i}}{2\sqrt{\mathbf{F}\cdot\mathbf{F}}} = \frac{\mathbf{F}\cdot\dfrac{\partial \mathbf{F}}{\partial x_i}}{|\mathbf{F}|}.$$

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In the context of the previous example we have:

$$\frac{\partial \phi}{\partial x} = \frac{km}{|\mathbf{r}-\mathbf{r}_0|^2}\cdot\frac{\partial}{\partial x}\left(-|\mathbf{r}-\mathbf{r}_0|\right) = \frac{-km}{|\mathbf{r}-\mathbf{r}_0|^2}\cdot\frac{(\mathbf{r}-\mathbf{r}_0)\cdot\mathbf{e}_1}{|\mathbf{r}-\mathbf{r}_0|} = \frac{-km(x-x_0)}{|\mathbf{r}-\mathbf{r}_0|^3}.$$

EXAMPLE

Show that the velocity field $\mathbf{v} = -\Omega y\,\mathbf{i} + \Omega x\,\mathbf{j}$ of rigid body rotation about the z-axis is **not conservative** if $\Omega \neq 0$.

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Solution: Method 1 (Direct approach). Suppose ϕ exists with $\nabla\phi = \mathbf{v}$. Then:

$$\frac{\partial \phi}{\partial x} = -\Omega y \quad \Longrightarrow \quad \phi(x, y) = -\Omega xy + C_1(y),$$

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Equating:

$$-\Omega xy + C_1(y) = \Omega xy + C_2(x) \quad\Longrightarrow\quad 2\Omega xy = C_1(y) - C_2(x).$$

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Equating:

$$-\Omega xy + C_1(y) = \Omega xy + C_2(x) \quad \Longrightarrow \quad 2\Omega xy = C_1(y) - C_2(x).$$

The left side depends on both x and y, while the right side separates. This is **impossible** for $\Omega \neq 0$!

EXAMPLE

Show that $\mathbf{v} = -\Omega y\,\mathbf{i} + \Omega x\,\mathbf{j}$ is not conservative if $\Omega \neq 0$.

Solution: **Method 2 (Mixed partials).** If $\mathbf{v}$ has a potential ϕ, then:

$$\frac{\partial \phi}{\partial x} = -\Omega y \quad \text{and} \quad \frac{\partial \phi}{\partial y} = \Omega x.$$

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Taking the mixed partial derivatives:

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The smoothness of $\mathbf{v}$ implies that ϕ should be smooth, so the mixed partials should be equal. But $-\Omega \neq \Omega$ when $\Omega \neq 0$.

Contradiction! Thus, no such ϕ can exist; $\mathbf{v}$ is not conservative.

Necessary condition for a conservative plane vector field

If $\mathbf{F}(x, y) = F_1(x, y)\,\mathbf{i} + F_2(x, y)\,\mathbf{j}$ is a conservative vector field in a domain D of the xy-plane, then the condition

$$\frac{\partial}{\partial y} F_1(x, y) = \frac{\partial}{\partial x} F_2(x, y)$$

must be satisfied at all points of D.

Necessary conditions for a conservative vector field in 3-space

If $\mathbf{F}(x, y, z) = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$ is a conservative vector field in a domain D in 3-space, then we must have, everywhere in D:

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \qquad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}.$$

Conservative Fields

Necessary Conditions

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Caution: These conditions are *necessary* but not always *sufficient*. Whether the converse holds depends on the topology of the domain D (e.g., whether D is simply connected).

Equipotential Surfaces and Curves

Equipotential Surfaces and Curves

If $\phi(x, y, z)$ is a potential function for the conservative vector field $\mathbf{F}$, then the level surfaces $\phi(x, y, z) = C$ of ϕ are called equipotential surfaces of $\mathbf{F}$.

Similarly, for a conservative plane vector field, the level curves of the potential function are called equipotential curves of the vector field.

Equipotential curves/surfaces are the **orthogonal trajectories** of the field lines; that is, they intersect the field lines at right angles.

EXAMPLE

Show that the vector field $\mathbf{F}(x, y) = x\,\mathbf{i} - y\,\mathbf{j}$ is conservative and find a potential function for it. Describe the field lines and the equipotential curves.

EXAMPLE (first part)

Show that $\mathbf{F}(x,y) = x\,\mathbf{i} - y\,\mathbf{j}$ is conservative and find a potential.

Solution: Since $\dfrac{\partial F_1}{\partial y} = 0 = \dfrac{\partial F_2}{\partial x}$ everywhere in $\mathbb{R}^2$, we would expect $\mathbf{F}$ to be conservative. Any potential function ϕ must satisfy:

$$\frac{\partial \phi}{\partial x} = F_1 = x \qquad \text{and} \qquad \frac{\partial \phi}{\partial y} = F_2 = -y.$$

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$$\phi(x, y) = \int x\,dx = \tfrac{1}{2}x^2 + C_1(y) \longrightarrow -y = \frac{\partial \phi}{\partial y} = C_1'(y)$$

Equipotential Surfaces and Curves

Example — Finding the Potential

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Thus, $\mathbf{F}$ is conservative, and for any constant C_2,

$$\boxed{\phi(x,y) = \frac{x^2 - y^2}{2} + C_2}$$ is a potential function for $\mathbf{F}$.

EXAMPLE (second part)

Describe the field lines and equipotential curves of $\mathbf{F}(x, y) = x\,\mathbf{i} - y\,\mathbf{j}$.

Solution: The field lines of $\mathbf{F}$ satisfy:

$$\frac{dx}{x} = \frac{dy}{-y} \quad \Longrightarrow \quad \ln|x| = -\ln|y| + \ln C_3 \quad \Longrightarrow \quad xy = C_3.$$

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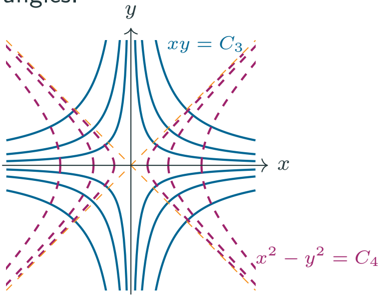

EXAMPLE

Decide whether the vector field

$$\mathbf{F} = \big(xy - \sin z\big)\,\mathbf{i} + \left(\tfrac{1}{2}x^2 - \tfrac{e^y}{z}\right)\mathbf{j} + \left(\tfrac{e^y}{z^2} - x\cos z\right)\mathbf{k}$$

is conservative in $D = \{(x, y, z) : z \neq 0\}$, and find a potential if it is.

EXAMPLE (continued)

$\mathbf{F} = \left(xy - \sin z\right)\mathbf{i} + \left(\frac{1}{2}x^2 - \frac{e^y}{z}\right)\mathbf{j} + \left(\frac{e^y}{z^2} - x\cos z\right)\mathbf{k}$

Solution: First, verify the necessary conditions:

$$\frac{\partial F_1}{\partial y} = x = \frac{\partial F_2}{\partial x}, \qquad \frac{\partial F_1}{\partial z} = -\cos z = \frac{\partial F_3}{\partial x}, \qquad \frac{\partial F_2}{\partial z} = \frac{e^y}{z^2} = \frac{\partial F_3}{\partial y}.$$

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All three conditions are satisfied. ✓

$\mathbf{F}$ *may* be conservative in domains not intersecting the xy-plane $z = 0$.

Solution: We need $\nabla\phi = \mathbf{F}$, i.e.:

$$\frac{\partial\phi}{\partial x} = xy - \sin z, \qquad \frac{\partial\phi}{\partial y} = \tfrac{1}{2}x^2 - \tfrac{e^y}{z}, \qquad \frac{\partial\phi}{\partial z} = \tfrac{e^y}{z^2} - x\cos z.$$

Equipotential Surfaces and Curves

A Three-Variable Example — Finding the Potential

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Step 1: Integrate the first equation with respect to x:

$$\phi(x,y,z) = \int (xy - \sin z)\,dx = \tfrac{1}{2}x^2 y - x\sin z + C_1(y,z).$$

Equipotential Surfaces and Curves

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Step 2: Differentiate with respect to y and compare:

$$\frac{\partial\phi}{\partial y} = \tfrac{1}{2}x^2 + \frac{\partial C_1}{\partial y} = \tfrac{1}{2}x^2 - \frac{e^y}{z} \quad\Longrightarrow\quad \frac{\partial C_1}{\partial y} = -\frac{e^y}{z}.$$

$$C_1(y,z) = -\frac{e^y}{z} + C_2(z).$$

Equipotential Surfaces and Curves

A Three-Variable Example — Finding the Potential (cont.)

Solution: So far: $\phi(x,y,z) = \frac{1}{2}x^2y - x\sin z - \dfrac{e^y}{z} + C_2(z)$.

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Step 3: Differentiate with respect to z and compare:

$$\frac{\partial \phi}{\partial z} = -x\cos z + \frac{e^y}{z^2} + C_2'(z) = \frac{e^y}{z^2} - x\cos z.$$

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Comparing: $C_2'(z) = 0$, hence $C_2(z) = C$ (const.).

Equipotential Surfaces and Curves

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$$\boxed{\phi(x,y,z) = \tfrac{1}{2}x^2y - x\sin z - \frac{e^y}{z} + C.}$$

Conservative Fields

EXAMPLE

For $(x, y) \neq (0, 0)$, define a vector field $\mathbf{F}(x, y)$ and a scalar field $\theta(x, y)$ as follows:

$$\mathbf{F}(x, y) = \frac{-y}{x^2 + y^2}\,\mathbf{i} + \frac{x}{x^2 + y^2}\,\mathbf{j},$$

$\theta(x, y) =$ the polar angle θ of (x, y) such that $0 < \theta < 2\pi$.

Thus, $x = r\cos\theta$ and $y = r\sin\theta$, where $r = \sqrt{x^2 + y^2}$. Verify:

(a) $\dfrac{\partial}{\partial y} F_1(x, y) = \dfrac{\partial}{\partial x} F_2(x, y)$ for $(x, y) \neq (0, 0)$.

(b) $\nabla\theta(x, y) = \mathbf{F}(x, y)$ for all $(x, y) \neq (0, 0)$ with $0 < \theta < 2\pi$.

(c) $\mathbf{F}$ is **not** conservative on the whole xy-plane excluding the origin.

EXAMPLE

$$\mathbf{F}(x, y) = \frac{-y}{x^2 + y^2}\,\mathbf{i} + \frac{x}{x^2 + y^2}\,\mathbf{j},$$

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Solution: **(a)** We have $F_1 = \dfrac{-y}{x^2 + y^2}$ and $F_2 = \dfrac{x}{x^2 + y^2}$. Thus:

$$\frac{\partial F_1}{\partial y} = \frac{-(x^2 + y^2) + y \cdot 2y}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2},$$

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$$\mathbf{F}(x,y) = \frac{-y}{x^2+y^2}\,\mathbf{i} + \frac{x}{x^2+y^2}\,\mathbf{j},$$

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$$\frac{\partial F_1}{\partial y} = \frac{-(x^2+y^2) + y \cdot 2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2},$$

$$\frac{\partial F_2}{\partial x} = \frac{(x^2+y^2) - x \cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

Therefore $\dfrac{\partial F_1}{\partial y} = \dfrac{\partial F_2}{\partial x}$ for all $(x,y) \neq (0,0)$. ✓

Solution: **(b)** Differentiate $x = r\cos\theta$ and $y = r\sin\theta$ implicitly with respect to x:

$$\left.\begin{aligned} 1 &= \frac{\partial r}{\partial x}\cos\theta - r\sin\theta\,\frac{\partial\theta}{\partial x}, \\ 0 &= \frac{\partial r}{\partial x}\sin\theta + r\cos\theta\,\frac{\partial\theta}{\partial x}. \end{aligned}\right\}$$

Conservative Fields

A Topological Obstruction — Part (b)

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Similarly, differentiation with respect to y produces:

$$\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r} = \frac{x/r}{r} = \frac{x}{x^2+y^2} = F_2. \quad \checkmark$$

Conservative Fields

A Topological Obstruction — Part (b)

Solution: **(b)** Differentiate $x = r\cos\theta$ and $y = r\sin\theta$ implicitly with respect to x:

$$\left.\begin{aligned} 1 &= \frac{\partial r}{\partial x}\cos\theta - r\sin\theta\,\frac{\partial \theta}{\partial x}, \\ 0 &= \frac{\partial r}{\partial x}\sin\theta + r\cos\theta\,\frac{\partial \theta}{\partial x}. \end{aligned}\right\} \to \frac{\partial \theta}{\partial x} = \frac{-\sin\theta}{r} = \frac{-y/r}{r} = \frac{-y}{x^2+y^2} = F_1.$$

Similarly, differentiation with respect to y produces:

$$\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r} = \frac{x/r}{r} = \frac{x}{x^2+y^2} = F_2. \quad \checkmark$$

These formulas hold only if $0 < \theta < 2\pi$; θ is **not even continuous** on the positive x-axis:

$$\lim_{y\to 0^+} \theta(x,y) = 0 \quad \text{but} \quad \lim_{y\to 0^-} \theta(x,y) = 2\pi \quad \text{for } x > 0.$$

Conservative Fields

A Topological Obstruction — Part (c)

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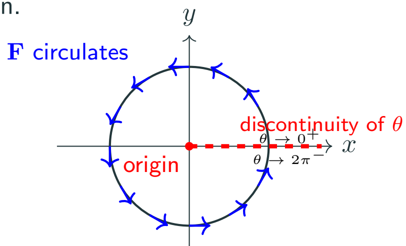

Line Integrals

Let C be a bounded, continuous parametric curve in $\mathbb{R}^3$. Recall that C is a smooth curve if it has a parametrization of the form

$$\mathbf{r} = \mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}, \qquad t \in I,$$

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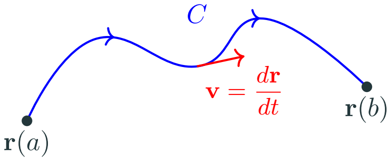
We will call C a smooth arc if it is a smooth curve with *finite* parameter interval $I = [a, b]$.



Line Integrals

Definition

Definition (Line Integral)

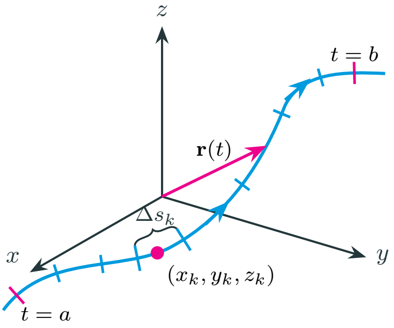
If f is defined on a curve C given parametrically by

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k},\ a \leq t \leq b,$$

then the line integral of f over C is

$$\int_C f(x, y, z)\,ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\,\Delta s_k,$$

provided this limit exists.



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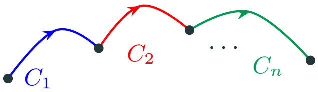
It also exists (for continuous f) if C is a piecewise smooth curve consisting of finitely many smooth arcs $C_1, \ldots, C_n$ linked end to end; in this case:

$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \cdots + \int_{C_n} f\,ds.$$



Recall that the **length** of C was evaluated by expressing the arc length element $ds = |d\mathbf{r}/dt|\,dt$ in terms of a parametrization $\mathbf{r} = \mathbf{r}(t)$, $(a \le t \le b)$:

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More general line integrals are evaluated similarly:

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Of course, all of the above discussion applies equally well to line integrals of functions $f(x, y)$ along curves C in the xy-plane.

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If $\mathbf{r} = \mathbf{r}^*(u)$, $\alpha \le u \le \beta$, is another parametrization of the same smooth curve C, then any point $\mathbf{r}(t)$ on C can be expressed as $\mathbf{r}^*(u)$ where $u = u(t)$.

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If $\mathbf{r}^*(u)$ traces C in the **same direction** as $\mathbf{r}(t)$, then $u(a) = \alpha$, $u(b) = \beta$, and $du/dt \ge 0$. If $\mathbf{r}^*(u)$ traces C in the **opposite direction**, then $u(a) = \beta$, $u(b) = \alpha$, and $du/dt \le 0$.

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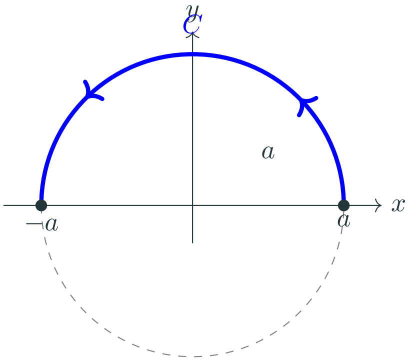
In either event:

$$\int_a^b f\big(\mathbf{r}(t)\big)\left|\frac{d\mathbf{r}}{dt}\right| dt = \int_a^b f\big(\mathbf{r}^*(u(t))\big)\left|\frac{d\mathbf{r}^*}{du}\frac{du}{dt}\right| dt = \int_\alpha^\beta f\big(\mathbf{r}^*(u)\big)\left|\frac{d\mathbf{r}^*}{du}\right| du.$$

Thus, the line integral is independent of the choice of parametrization.

EXAMPLE

A circle of radius $a > 0$ has centre at the origin in the xy-plane. Let C be the half of this circle lying in the half-plane $y \geq 0$. Use two different parametrizations of C to find the moment of C about $y = 0$.



Solution: We are asked to calculate $\int_C y\, ds$.

Parametrization 1: $\mathbf{r} = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j}, \quad 0 \le t \le \pi.$

Line Integrals

Example — Parametrization 1

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$$\frac{d\mathbf{r}}{dt} = -a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j} \quad \Longrightarrow \quad \left|\frac{d\mathbf{r}}{dt}\right| = a.$$

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The moment of C about $y = 0$ is:

$$\int_C y\,ds = \int_0^\pi a\sin t \cdot a\,dt = a^2\int_0^\pi \sin t\,dt = a^2\big[-\cos t\big]_0^\pi = \boxed{2a^2.}$$

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Thus, the moment of C about $y = 0$ is:

$$\int_C y\,ds = \int_{-a}^{a} \sqrt{a^2 - x^2} \cdot \frac{a}{\sqrt{a^2 - x^2}}\,dx = a\int_{-a}^{a} dx = \boxed{2a^2.}$$

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Both parametrizations give the same result, confirming that the line integral is **independent of parametrization**.