MAT124 MATHEMATICS II

An Application of Multiple Integrals - Surface Area of a Graph, Vector Fields

Outline

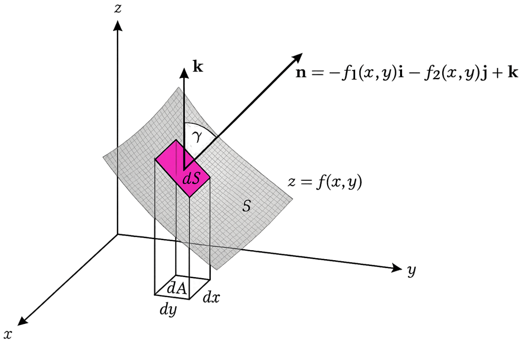
An Application of Multiple Integrals

- Surface Area of a Graph

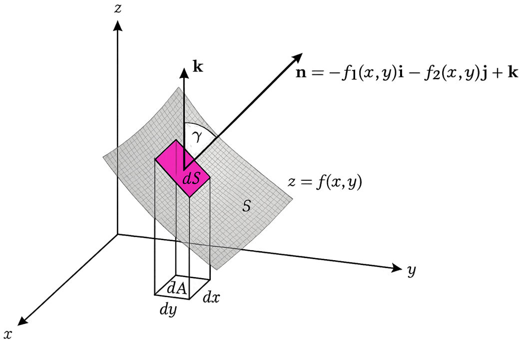
Vector Fields

An Application of Multiple Integrals

Surface Area of a Graph



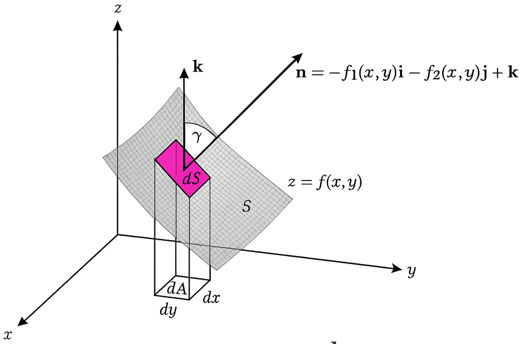
Surface Area of a Graph



$dS =$ the vertical projection of dA on the tangent plane to the surface $z = f(x,y)$

$= \sec\gamma \cdot dA$

Surface Area of a Graph



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\begin{aligned}
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&\quad\ \text{on the tangent plane to the} \\
&\quad\ \text{surface } z = f(x, y) \\
&= \sec\gamma \cdot dA
\end{aligned}
$$

$$\cos\gamma = \frac{\mathbf{n}\cdot\mathbf{k}}{|\mathbf{n}||\mathbf{k}|} = \frac{1}{\sqrt{1 + (f_1(x,y))^2 + (f_2(x,y))^2}},$$

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA.$$

Surface Area of a Graph

Therefore the surface is calculated by the following formula:

Surface Area Formula by Double Integral

$$S = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA.$$

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$$dS = \sqrt{1 + 4x^2 + 4y^2}\, dA = \sqrt{1 + 4r^2}\, r\, dr\, d\theta.$$

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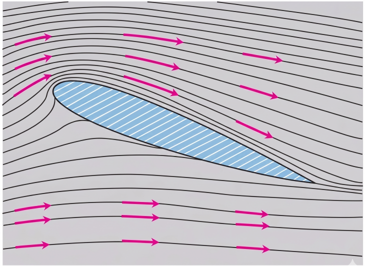
Let $u = 1 + 4r^2$

$$= (2\pi)\frac{1}{8} \int_1^{1+4a^2} \sqrt{u}\, du$$

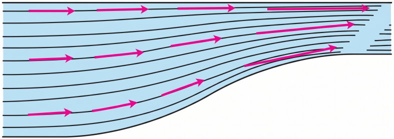
$$= \frac{\pi}{4}\left(\frac{2}{3}\right) u^{3/2}\Big|_1^{1+4a^2} = \frac{\pi}{6}\left((1 + 4a^2)^{3/2} - 1\right) \text{ square units.}$$

Vector Fields

Vector and Scalar Fields



Velocity vectors of a flow around an airfoil in a wind tunnel.



Streamlines in a contracting channel. The water speeds up as the channel narrows and the velocity vectors increase in length.

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$$\mathbf{F}(x, y, z) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}.$$

(Note that the subscripts here represent components of a vector, not partial derivatives.)

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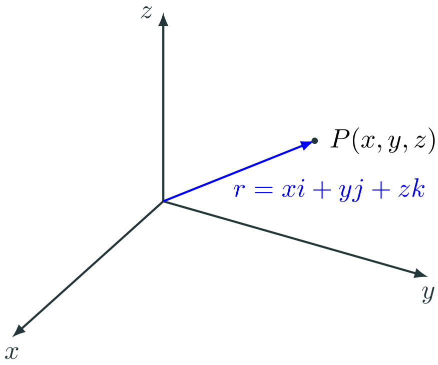
If $F_3(x, y, z) = 0$ and F_1 and F_2 are independent of z, then $\mathbf{F}$ reduces to

$$\mathbf{F}(x, y) = F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}$$

and so is called a plane vector field, or a vector field in the xy-plane.

Vector and Scalar Fields



Using position vectors, we may write $\mathbf{F}(\mathbf{r})$ as a shorthand for $\mathbf{F}(x, y, z)$.

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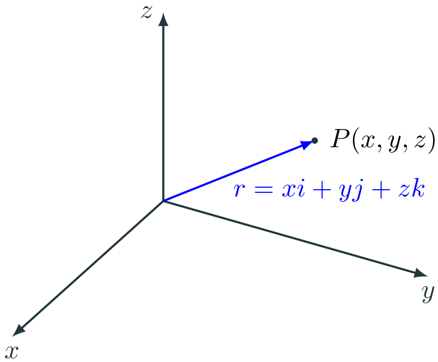

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In the context of discussion of vector fields, a scalar-valued function of a vector variable is called a scalar field.

Vector and Scalar Fields

Many of the results we prove about vector fields require that the field be smooth in some sense. We will call a vector field smooth wherever its component scalar fields have continuous partial derivatives of all orders. (For most purposes, however, second order would be sufficient.)

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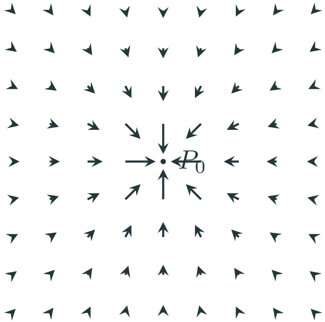
(**The gravitational field of a point mass**) The gravitational force field due to a point mass m located at point P_0 having position vector $\mathbf{r}_0$ is

$$\mathbf{F}(x,y,z) = \mathbf{F}(\mathbf{r}) = \frac{-km}{|\mathbf{r}-\mathbf{r}_0|^3}(\mathbf{r}-\mathbf{r}_0)$$

$$= \frac{-km\left[(x-x_0)\mathbf{i}+(y-y_0)\mathbf{j}+(z-z_0)\mathbf{k}\right]}{\left[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right]^{3/2}},$$

where $k>0$ is a constant. $\mathbf{F}$ points toward the point $\mathbf{r}_0$ and has magnitude

$$|\mathbf{F}| = km/|\mathbf{r}-\mathbf{r}_0|^2.$$



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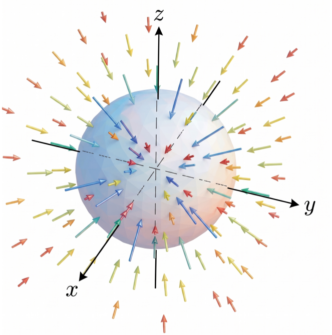
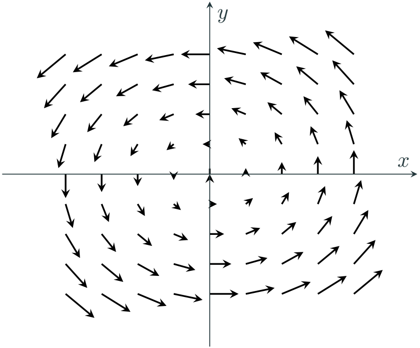


EXAMPLE

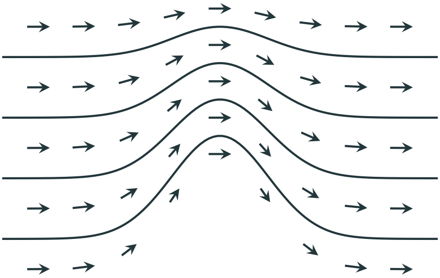
The velocity field of a solid rotating about the z-axis with angular velocity $\mathbf{\Omega} = \Omega\mathbf{k}$ is

$$\mathbf{v}(x, y, z) = \mathbf{v}(\mathbf{r}) = \mathbf{\Omega} \times \mathbf{r} = -\Omega y\mathbf{i} + \Omega x\mathbf{j}.$$

Being the same in all planes normal to the z-axis, $\mathbf{v}$ can be regarded as a plane vector field.



Field Lines (Integral Curves, Trajectories, Streamlines)



<u>Question:</u> What path will be followed by a particle whose velocity is given by a vector field?

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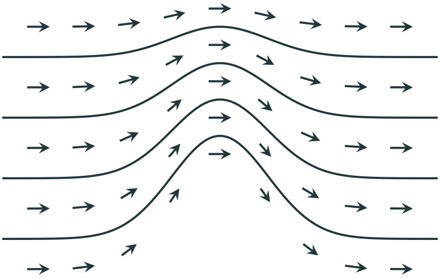

Question: What path will be followed by a particle whose velocity is given by a vector field?

Answer: The path will be a curve to which the field is tangent at every point.

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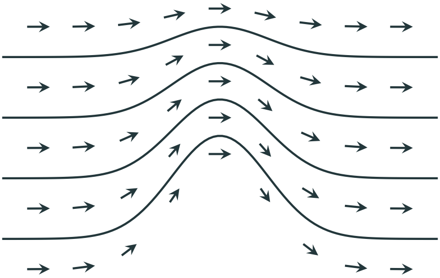

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Such curves are field lines for the given vector field.

Field Lines (Integral Curves, Trajectories, Streamlines)

The field lines of $\mathbf{F}$ do not depend on the magnitude of $\mathbf{F}$ at any point but only on the direction of the field. If the field line through some point has parametric equation $\mathbf{r} = \mathbf{r}(t)$, then its tangent vector $d\mathbf{r}/dt$ must be parallel to $\mathbf{F}(\mathbf{r}(t))$ for all t. Thus

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$$\frac{d\mathbf{r}}{dt} = \lambda(t)\mathbf{F}(\mathbf{r}(t)).$$

If we break the equation into components,

$$\frac{dx}{dt} = \lambda(t)\, F_1(x, y, z), \qquad \frac{dy}{dt} = \lambda(t)\, F_2(x, y, z), \qquad \frac{dz}{dt} = \lambda(t)\, F_3(x, y, z),$$

we can obtain equivalent differential expressions for $\lambda(t)\, dt$ and hence write the differential equation for the field lines in the form

$$\frac{dx}{F_1(x, y, z)} = \frac{dy}{F_2(x, y, z)} = \frac{dz}{F_3(x, y, z)}.$$

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Find the field lines of the gravitational force field

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Integrating all three expressions leads to

$$\ln|x-x_0| + \ln C_1 = \ln|y-y_0| + \ln C_2 = \ln|z-z_0| + \ln C_3,$$

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$$\boxed{C_1(x-x_0) = C_2(y-y_0) = C_3(z-z_0).}$$

straight lines through (x_0, y_0, z_0)

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Find the field lines of the velocity field $\mathbf{v} = \Omega(-y\mathbf{i} + x\mathbf{j})$.

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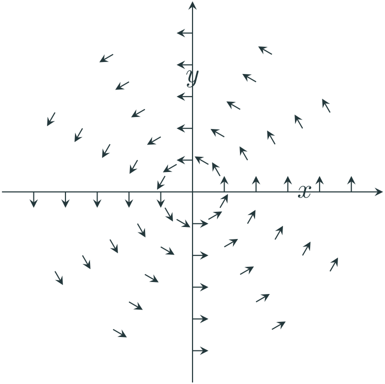
Solution: The field lines satisfy the differential equation $\frac{dx}{-y} = \frac{dy}{x}$.

We can separate variables in this equation to get $x\,dx = -y\,dy$. Integration then gives $x^2/2 = -y^2/2 + C/2$, or $x^2 + y^2 = C$.



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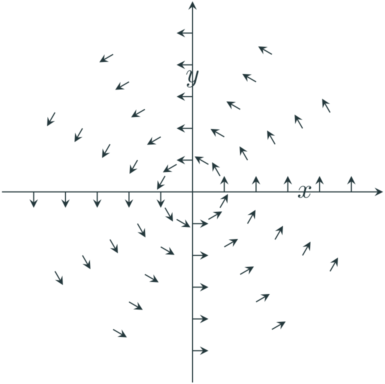

If we regard $\mathbf{v}$ as a vector field in 3-space, we find that the field lines are horizontal circles centred on the z-axis:

$$x^2 + y^2 = C_1, \qquad z = C_2.$$

Vector Fields in Polar Coordinates

A vector field in the plane can be expressed in terms of polar coordinates in the form

$$\mathbf{F} = \mathbf{F}(r, \theta) = F_r(r, \theta)\, \hat{\mathbf{r}} + F_\theta(r, \theta)\, \hat{\boldsymbol{\theta}},$$

where $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$, defined everywhere except at the origin by

$$\begin{aligned} \hat{\mathbf{r}} &= \cos\theta\, \mathbf{i} + \sin\theta\, \mathbf{j} \\ \hat{\boldsymbol{\theta}} &= -\sin\theta\, \mathbf{i} + \cos\theta\, \mathbf{j}, \end{aligned}$$

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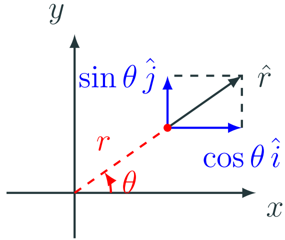
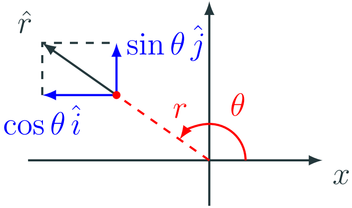


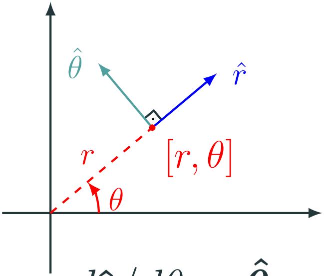


Vector Fields in Polar Coordinates

For the vector field

$$\mathbf{F} = \mathbf{F}(r,\theta) = \boxed{F_r(r,\theta)\,\hat{\mathbf{r}}} + \boxed{F_\theta(r,\theta)\,\hat{\boldsymbol{\theta}}},$$

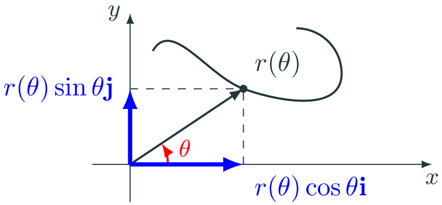
$F_r(r,\theta)$ is called the radial component of $\mathbf{F}$, and $F_\theta(r,\theta)$ is called the transverse component.



$$d\hat{\mathbf{r}}/d\theta = \hat{\boldsymbol{\theta}}$$

Vector Fields in Polar Coordinates

A curve with polar equation $r = r(\theta)$ can be expressed in vector parametric form, $\mathbf{r} = r\hat{\mathbf{r}}$.



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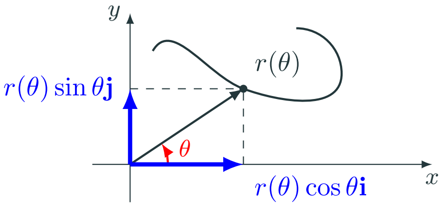

This curve is a field line of $\mathbf{F}$ if its differential tangent vector

$$d\mathbf{r} = dr\,\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{d\theta}d\theta = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}}$$

is parallel to the field vector $\mathbf{F}(r,\theta)$ at any point except the origin, that is, if $r = f(\theta)$ satisfies the differential equation

$$\frac{dr}{F_r(r,\theta)} = \frac{r\,d\theta}{F_\theta(r,\theta)}.$$

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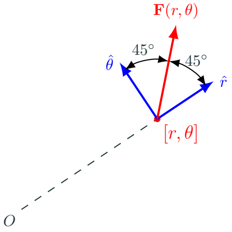

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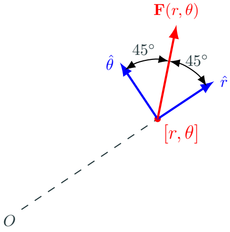

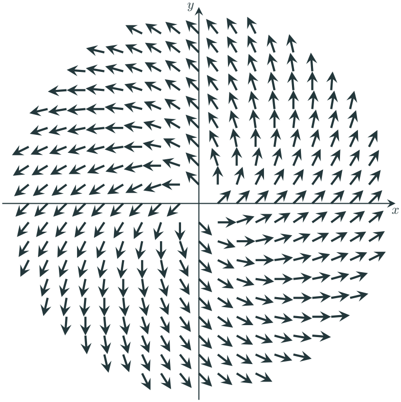

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