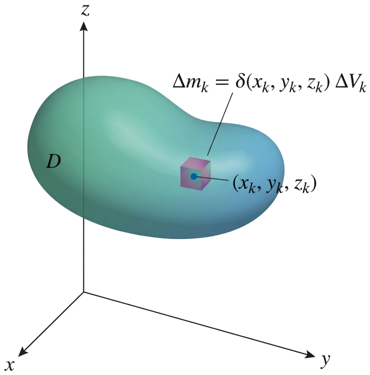
MAT124 MATHEMATICS II

An Application of Multiple Integrals, Integration in Cylindrical Coordinates, Integration in Spherical Coordinates, Change of Variables in Triple Integrals

Outline

An Application of Multiple Integrals

Masses and First Moments



If $\delta(x, y, z)$ is the density (mass per unit volume) of an object occupying a region D in space, the integral of δ over D gives the mass of the object. To see why, imagine partitioning the object into n mass elements.

The object's mass is the limit

$$\begin{aligned} M &= \lim_{n\to\infty} \sum_{k=1}^{n} \Delta m_k \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \delta(x_k, y_k, z_k)\,\Delta V_k \\ &= \iiint_D \delta(x, y, z)\,dV. \end{aligned}$$

Masses and First Moments

The first moment of a solid region D about a coordinate plane is defined as the triple integral over D of the distance from a point (x, y, z) in D to the plane multiplied by the density of the solid at that point. For instance, the first moment about the yz-plane is the integral

$$M_{yz} = \iiint\limits_{D} x\delta(x, y, z)\, dV.$$

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The center of mass is found from the first moments. For instance, the x-coordinate of the center of mass is $\bar{x} = M_{yz}/M$.

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For a two-dimensional object, such as a thin, flat plate, we calculate first moments about the coordinate axes by simply dropping the z-coordinate. So the first moment about the y-axis is the double integral over the region R forming the plate of the distance from the axis multiplied by the density, or

$$M_y = \iint_R x\delta(x, y)\, dA.$$

Masses and First Moments

Three-dimensional Solid

THREE-DIMENSIONAL SOLID

Mass: Let $\delta = \delta(x, y, z)$ be the density at (x, y, z)

$$M = \iiint\limits_{D} \delta \, dV$$

First moments about the coordinate planes:

$$M_{yz} = \iiint\limits_{D} x \, \delta \, dV, \qquad M_{xz} = \iiint\limits_{D} y \, \delta \, dV, \qquad M_{xy} = \iiint\limits_{D} z \, \delta \, dV$$

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}, \qquad \bar{y} = \frac{M_{xz}}{M}, \qquad \bar{z} = \frac{M_{xy}}{M}$$

Masses and First Moments

Two-dimensional Plate

TWO-DIMENSIONAL PLATE

Mass: Let $\delta = \delta(x, y)$ be the density at (x, y)

$$M = \iint_R \delta \, dA$$

First moments:

$$M_y = \iint_R x \, \delta \, dA, \qquad M_x = \iint_R y \, \delta \, dA$$

Center of mass:

$$\bar{x} = \frac{M_y}{M}, \qquad \bar{y} = \frac{M_x}{M}$$

Masses and First Moments

Example

EXAMPLE

Find the center of mass of a solid of constant density δ bounded below by the disk $R : x^2 + y^2 \leq 4$ in the plane $z = 0$ and above by the paraboloid $z = 4 - x^2 - y^2$.

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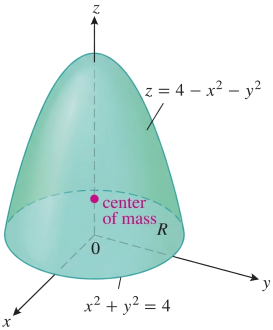

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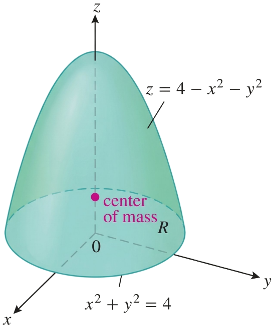

By symmetry $\bar{x} = \bar{y} = 0$. To find $\bar{z}$,

$$
\begin{aligned}
M_{xy} &= \iiint_R z\,\delta\,dz\,dy\,dx = \iint_R \left[\frac{z^2}{2}\right]_{z=0}^{z=4-x^2-y^2} \delta\,dy\,dx \\
&= \frac{\delta}{2}\iint_R (4-x^2-y^2)^2\,dy\,dx \\
&= \frac{\delta}{2}\int_0^{2\pi}\int_0^2 (4-r^2)^2 r\,dr\,d\theta \\
&= \frac{\delta}{2}\int_0^{2\pi}\left[-\frac{1}{6}(4-r^2)^3\right]_{r=0}^{r=2} d\theta \\
&= \frac{16\delta}{3}\int_0^{2\pi} d\theta = \frac{32\pi\delta}{3}.
\end{aligned}
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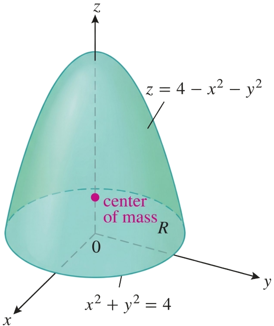

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$$\begin{aligned} M_{xy} &= \iiint\limits_R z\,\delta\,dz\,dy\,dx = \iint\limits_R \left[\frac{z^2}{2}\right]_{z=0}^{z=4-x^2-y^2} \delta\,dy\,dx \\ &= \frac{\delta}{2}\iint\limits_R (4 - x^2 - y^2)^2\,dy\,dx \\ &= \frac{\delta}{2}\int_0^{2\pi}\int_0^2 (4 - r^2)^2 r\,dr\,d\theta \\ &= \frac{\delta}{2}\int_0^{2\pi}\left[-\frac{1}{6}(4 - r^2)^3\right]_{r=0}^{r=2} d\theta \\ &= \frac{16\delta}{3}\int_0^{2\pi} d\theta = \frac{32\pi\delta}{3}. \end{aligned}$$

$$M = \iint\limits_R \int_0^{4-x^2-y^2} \delta\,dz\,dy\,dx = 8\pi\delta \quad \rightarrow \quad \bar{z} = (M_{xy}/M) = 4/3,\ (\bar{x}, \bar{y}, \bar{z}) = \boxed{(0,\ 0,\ 4/3)}$$

Masses and First Moments

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Find the centroid of the region in the first quadrant that is bounded above by the line $y = x$ and below by the parabola $y = x^2$.

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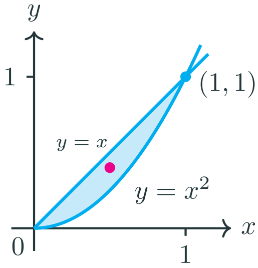

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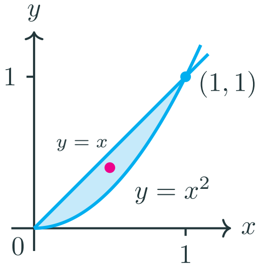

$$M = \int_0^1 \int_{x^2}^{x} 1\, dy\, dx = \int_0^1 [y]_{y=x^2}^{y=x}\, dx$$

$$= \int_0^1 (x - x^2) dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

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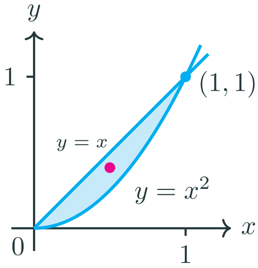

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$$M_x = \int_0^1 \int_{x^2}^{x} y \, dy \, dx = \int_0^1 \left[\frac{y^2}{2} \right]_{y=x^2}^{y=x} dx$$

$$= \int_0^1 \left(\frac{x^2}{2} - \frac{x^4}{2} \right) dx = \int_0^1 \left(\frac{x^2 - x^4}{2} \right) dx$$

$$= \left[\frac{x^3}{6} - \frac{x^5}{10} \right]_0^1 = \frac{1}{6} - \frac{1}{10} = \frac{5-3}{30} = \frac{2}{30} = \frac{1}{15}$$

Masses and First Moments

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Solution:

$$\begin{aligned} M_y &= \int_0^1 \int_{x^2}^{x} x\,dy\,dx = \int_0^1 [xy]_{y=x^2}^{y=x}\,dx \\ &= \int_0^1 (x^2 - x^3)dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 \\ &= \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \end{aligned}$$

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$$\bar{x} = \frac{M_y}{M} = \frac{1/12}{1/6} = \frac{1}{2}, \quad \bar{y} = \frac{M_x}{M} = \frac{1/15}{1/6} = \frac{2}{5}$$

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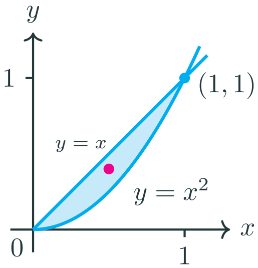
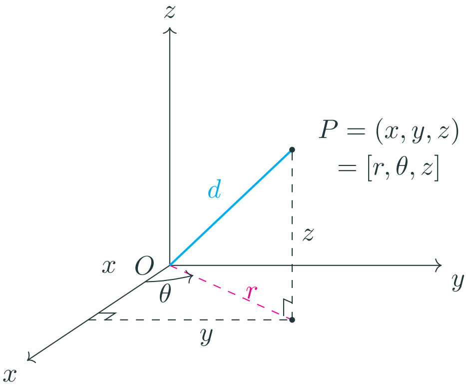


The centroid is the point $(1/2, 2/5)$.

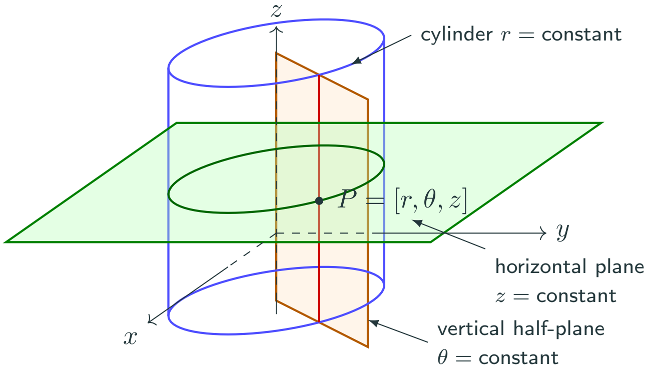
Change of Variables in Triple Integrals

Integration in Cylindrical Coordinates



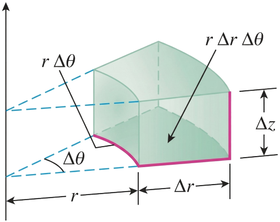
$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z,$$
$$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}$$

Integration in Cylindrical Coordinates



Integration in Cylindrical Coordinates

When computing triple integrals over a region D in cylindrical coordinates, we partition the region into n small cylindrical wedges, rather than into rectangular boxes.



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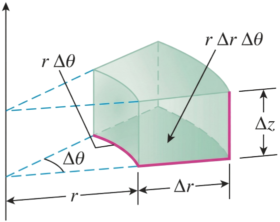

For a point (r_k, θ_k, z_k) in the center of the kth wedge, we calculate in polar coordinates that $\Delta A_k = r_k \Delta r_k \Delta\theta_k$. So $\Delta V_k = \Delta z_k\, r_k\, \Delta r_k\, \Delta\theta_k$ and a Riemann sum for f over D has the form

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k, z_k)\, \Delta z_k\, r_k\, \Delta r_k\, \Delta\theta_k.$$

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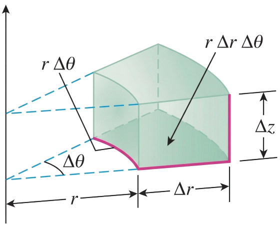

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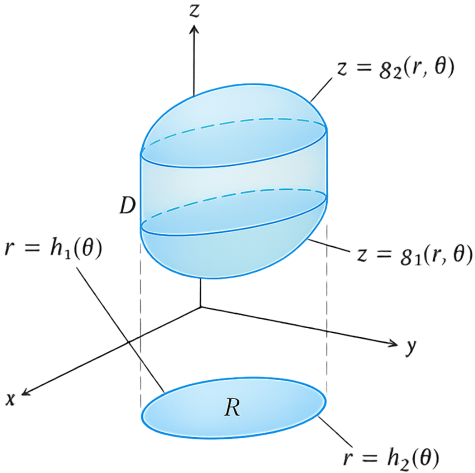
$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k, z_k)\, \Delta z_k\, r_k\, \Delta r_k\, \Delta\theta_k.$$

$$\lim_{n\to\infty} S_n = \iiint_D f\, dV = \iiint_D f\, dz\, r\, dr\, d\theta.$$

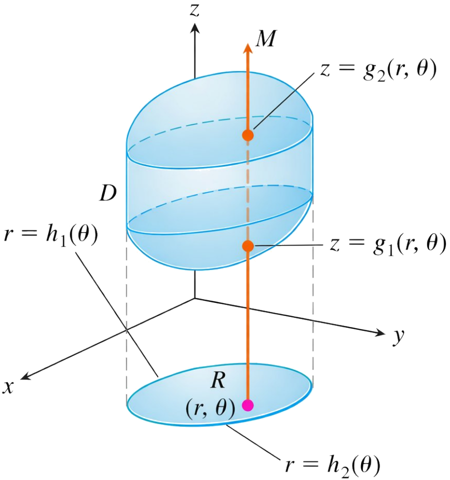
How to Integrate in Cylindrical Coordinates

1. **Sketch:** Sketch the region D along with its projection R on the xy-plane. Label the surfaces and curves that bound D and R.



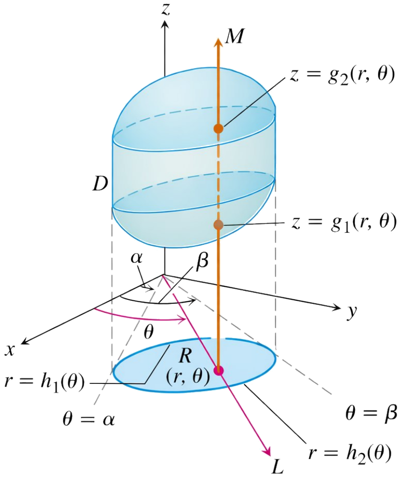
How to Integrate in Cylindrical Coordinates

2. **Find the z-limits of integration:** Draw a line M through a typical point (r, θ) of R parallel to the z-axis. As z increases, M enters D at $z = g_1(r, \theta)$ and leaves at $z = g_2(r, \theta)$. These are the z-limits of integration.



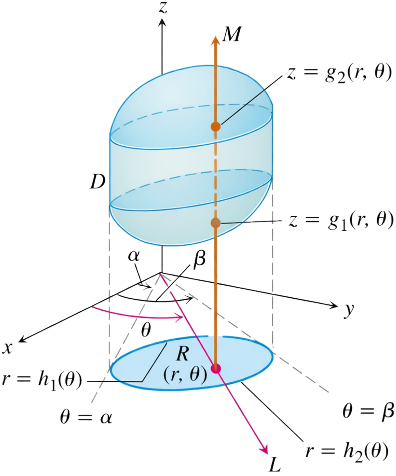
How to Integrate in Cylindrical Coordinates

3. **Find the r-limits of integration:** Draw a ray L through (r, θ) from the origin. The ray enters R at $r = h_1(\theta)$ and leaves at $r = h_2(\theta)$. These are the r-limits of integration.



How to Integrate in Cylindrical Coordinates

4. **Find the θ-limits of integration:** As L sweeps across R, the angle θ it makes with the positive x-axis runs from $\theta = \alpha$ to $\theta = \beta$. These are the θ-limits of integration.



How to Integrate in Cylindrical Coordinates

5. **Writing triple integral:** Finally, construct the corresponding triple integral.

$$\iiint\limits_D f(r,\theta,z)\,dV$$

$$= \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r,\theta,z)\,dz\,r\,dr\,d\theta.$$

How to Integrate in Cylindrical Coordinates

Example

EXAMPLE

Find the centroid $(\delta = 1)$ of the solid enclosed by the cylinder $x^2 + y^2 = 4$, bounded above by the paraboloid $z = x^2 + y^2$, and bounded below by the xy-plane.

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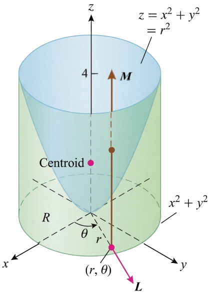

The solid's centroid $(\bar{x}, \bar{y}, \bar{z})$ lies on its axis of symmetry, here the z-axis. This makes $\bar{x} = \bar{y} = 0$. To find $\bar{z}$, we divide the first moment M_{xy} by the mass M.

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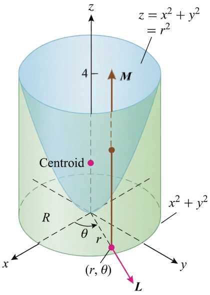

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$$M_{xy} = \int_0^{2\pi}\int_0^{2}\int_0^{r^2} z\,dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^{2}\left[\frac{z^2}{2}\right]_0^{r^2} r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2}\frac{r^5}{2}\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^6}{12}\right]_0^{2} d\theta = \int_0^{2\pi}\frac{16}{3}\,d\theta = \frac{32\pi}{3}.$$

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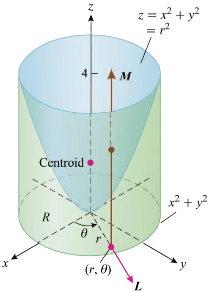

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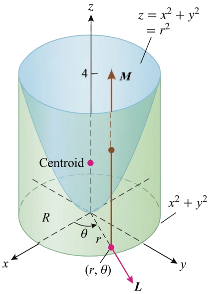

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$$\boxed{\bar{z} = \frac{M_{xy}}{M} = \frac{32\pi}{3}\cdot\frac{1}{8\pi} = \frac{4}{3}}$$

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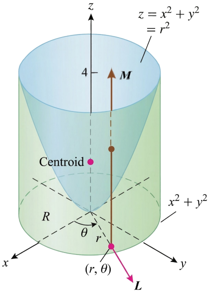

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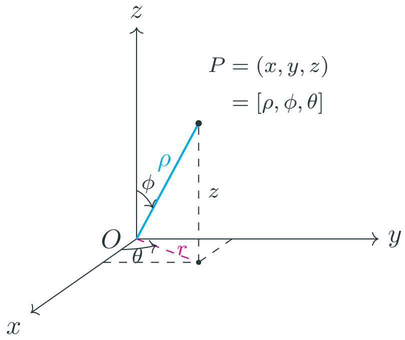
$$\boxed{\bar{z} = \frac{M_{xy}}{M} = \frac{32\pi}{3} \cdot \frac{1}{8\pi} = \frac{4}{3}}$$

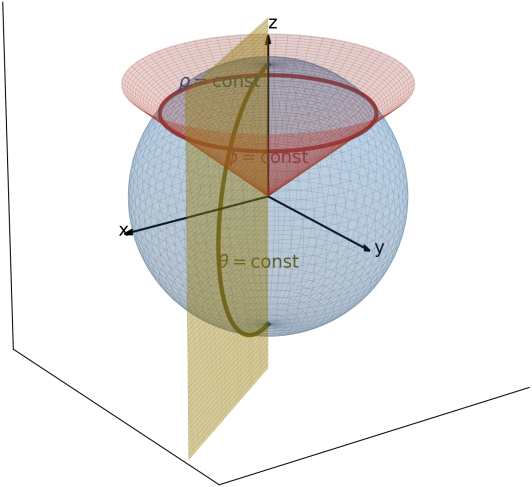
$$\boxed{\text{Centroid} = (0, 0, \frac{4}{3}).}$$

Integration in Spherical Coordinates

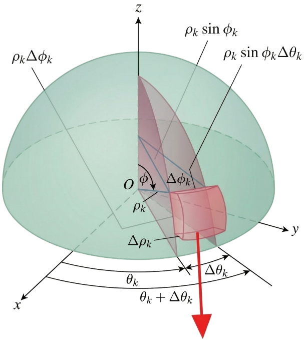


$$r = \rho \sin \phi, \qquad x = r \cos \theta = \rho \sin \phi \cos \theta,$$
$$z = \rho \cos \phi, \qquad y = r \sin \theta = \rho \sin \phi \sin \theta,$$
$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}.$$

Integration in Spherical Coordinates



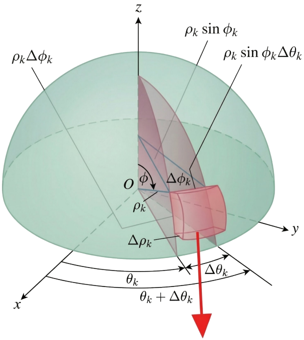
Integration in Spherical Coordinates



$$\Delta V_k = \rho_k^2 \sin \phi_k \Delta\rho_k \Delta\phi_k \Delta\theta_k$$

When computing triple integrals over a region D in spherical coordinates, we partition the region into n spherical wedges.

Integration in Spherical Coordinates

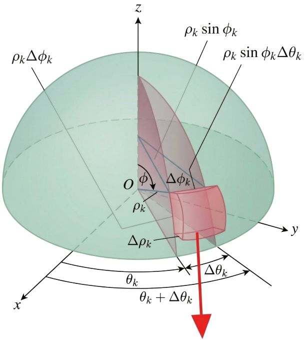


$$\Delta V_k = \rho_k^2 \sin \phi_k \Delta\rho_k \Delta\phi_k \Delta\theta_k$$

When computing triple integrals over a region D in spherical coordinates, we partition the region into n spherical wedges. The corresponding Riemann sum for a function $f(\rho, \phi, \theta)$ is

$$S_n = \sum_{k=1}^{n} f(\rho_k, \phi_k, \theta_k)\, \rho_k^2 \sin \phi_k \, \Delta\rho_k \, \Delta\phi_k \, \Delta\theta_k.$$

Integration in Spherical Coordinates



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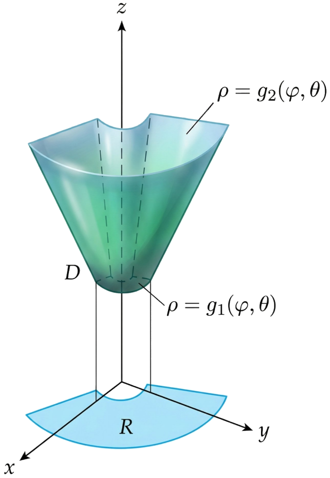
$$\lim_{n\to\infty} S_n = \iiint_D f(\rho, \phi, \theta)\, dV$$

$$= \iiint_D f(\rho, \phi, \theta)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

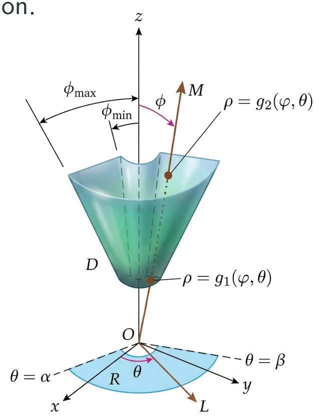
How to Integrate in Spherical Coordinates

1. **Sketch:** Sketch the region D along with its projection R on the xy-plane. Label the surfaces that bound D.



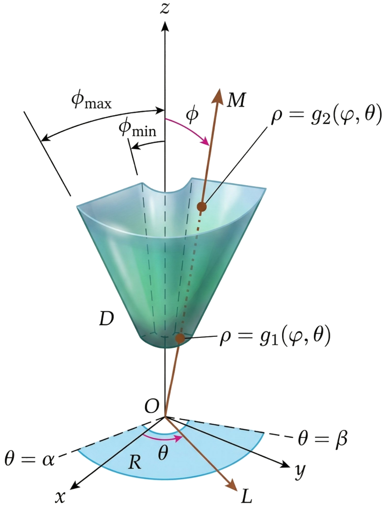
How to Integrate in Spherical Coordinates

2. **Find the ρ-limits of integration:** Draw a ray M from the origin through D making an angle ϕ with the positive z-axis. Also draw the projection of M on the xy-plane (call the projection L). The ray L makes an angle θ with the positive x-axis. As ρ increases, M enters D at $\rho = g_1(\phi, \theta)$ and leaves at $\rho = g_2(\phi, \theta)$. These are the ρ-limits of integration.



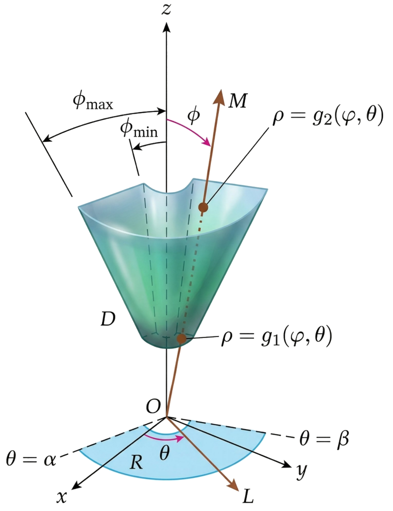
How to Integrate in Spherical Coordinates

3. **Find the ϕ-limits of integration:** For any given θ, the angle ϕ that M makes with the z-axis runs from $\phi = \phi_{\text{min}}$ to $\phi = \phi_{\text{max}}$. These are the ϕ-limits of integration.



How to Integrate in Spherical Coordinates

4. **Find the θ-limits of integration:** The ray L sweeps over R as θ runs from α to β. These are the θ-limits of integration.



How to Integrate in Spherical Coordinates

5. **Writing triple integral:** Finally, construct the corresponding triple integral.

$$\iiint\limits_{D} f(\rho, \phi, \theta)\, dV$$

$$= \int_{\theta=\alpha}^{\theta=\beta} \int_{\phi=\phi_{\min}}^{\phi=\phi_{\max}} \int_{\rho=g_1(\phi,\theta)}^{\rho=g_2(\phi,\theta)} f(\rho, \phi, \theta)\, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

How to Integrate in Spherical Coordinates

Example

EXAMPLE

Find the volume of the “ice cream cone” D cut from the solid sphere $\rho \leq 1$ by the cone $\phi = \pi/3$.

How to Integrate in Spherical Coordinates

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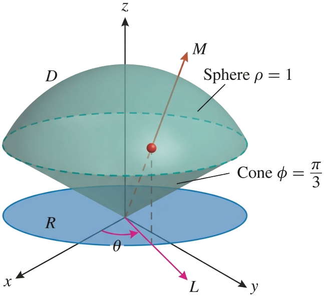
Find the volume of the "ice cream cone" D cut from the solid sphere $\rho \le 1$ by the cone $\phi = \pi/3$.

Solution:



How to Integrate in Spherical Coordinates

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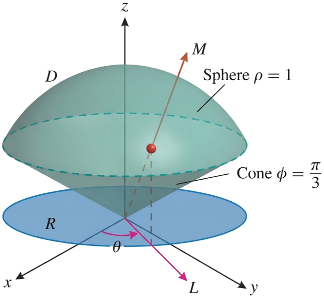


$$V = \iiint\limits_D \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/3} \left[\frac{\rho^3}{3}\right]_0^1 \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/3} \frac{1}{3} \sin\phi \, d\phi \, d\theta$$

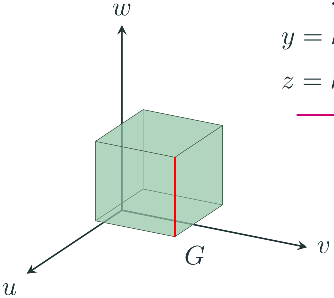
$$= \int_0^{2\pi} \left[-\frac{1}{3}\cos\phi\right]_0^{\pi/3} d\theta$$

$$= \int_0^{2\pi} \left(-\frac{1}{6} + \frac{1}{3}\right) d\theta = \frac{1}{6}(2\pi) = \frac{\pi}{3}.$$

Substitutions in Triple Integrals

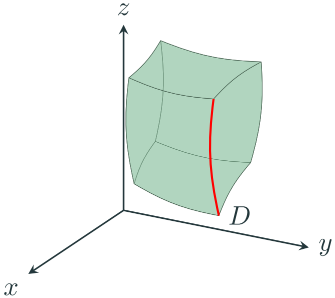
Suppose that a region G in uvw-space is transformed one-to-one into the region D in xyz-space by differentiable equations of the form

$$x = g(u,v,w), \qquad y = h(u,v,w), \qquad z = k(u,v,w).$$



Cartesian uvw-space

$$\xrightarrow{\begin{array}{l} x = g(u,v,w) \\ y = h(u,v,w) \\ z = k(u,v,w) \end{array}}$$



Cartesian xyz-space

Substitutions in Triple Integrals

Then any function $F(x, y, z)$ defined on D can be thought of as a function

$$F(g(u, v, w),\, h(u, v, w),\, k(u, v, w)) = H(u, v, w)$$

defined on G. If g, h, and k have continuous first partial derivatives, then the integral of $F(x, y, z)$ over D is related to the integral of $H(u, v, w)$ over G by the equation

$$\iiint\limits_{D} F(x, y, z)\, dx\, dy\, dz = \iiint\limits_{G} H(u, v, w)\ |J(u, v, w)|\, du\, dv\, dw,$$

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$$\iiint\limits_{D} F(x, y, z)\, dx\, dy\, dz = \iiint\limits_{G} H(u, v, w)\, |J(u, v, w)|\, du\, dv\, dw,$$

The factor $J(u, v, w)$ is the Jacobian determinant

$$J(u, v, w) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial(x, y, z)}{\partial(u, v, w)}.$$

Substitutions in Triple Integrals

Example

EXAMPLE

Evaluate

$$\int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left(\frac{2x-y}{2} + \frac{z}{3} \right) dx\, dy\, dz$$

by applying the transformation $u = (2x-y)/2, \quad v = y/2, \quad w = z/3$ and integrating over an appropriate region in uvw-space.

Substitutions in Triple Integrals

Example

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$$\int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left(\frac{2x-y}{2} + \frac{z}{3} \right) dx\, dy\, dz$$

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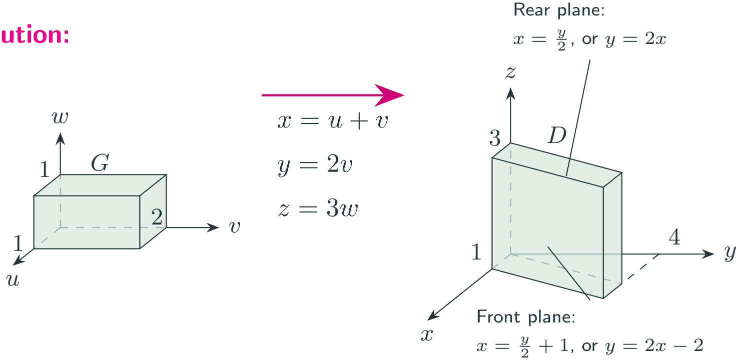
Solution:



$$x = u + v$$

$$y = 2v$$

$$z = 3w$$

Rear plane:

$x = \frac{y}{2}$, or $y = 2x$

Front plane:

$x = \frac{y}{2} + 1$, or $y = 2x - 2$

Substitutions in Triple Integrals

Example

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Evaluate

$$\int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left(\frac{2x-y}{2} + \frac{z}{3} \right) dx\, dy\, dz$$

by applying the transformation $u = (2x - y)/2, \quad v = y/2, \quad w = z/3$ and integrating over an appropriate region in uvw-space.

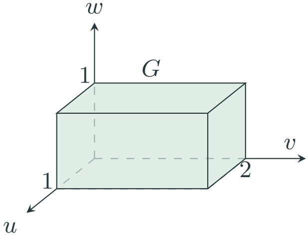
Solution: Since $x = u + v, \quad y = 2v \quad z = 3w$, we have

$$J(u, v, w) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6.$$

Substitutions in Triple Integrals

Example

Solution: Finally,



$$\int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left(\frac{2x-y}{2} + \frac{z}{3} \right) dx\,dy\,dz$$

$$= \int_0^1 \int_0^2 \int_0^1 (u+w)|J(u,v,w)|\,du\,dv\,dw$$

$$= \int_0^1 \int_0^2 \int_0^1 (u+w)(6)\,du\,dv\,dw$$

$$= 6 \int_0^1 \int_0^2 \left[\frac{u^2}{2} + uw \right]_0^1 dv\,dw$$

$$= 6 \int_0^1 \int_0^2 \left(\frac{1}{2} + w \right) dv\,dw$$

$$= 6 \int_0^1 \left[\frac{v}{2} + vw \right]_0^2 dw = 6 \int_0^1 (1+2w)\,dw = 12.$$