MAT124 MATHEMATICS II

Limits and Continuity
Partial Derivatives

Outline

Limits and Continuity

Partial Derivatives

Tangent Planes and Normal Lines

Higher-Order Derivatives

Limits and Continuity

Definition of Limit

We say that $\lim_{(x,y)\to(a,b)} f(x,y) = L$, provided that

(i) every neighbourhood of (a,b) contains points of the domain of f different from (a,b), and

(ii) for every positive number ϵ there exists a positive number $\delta = \delta(\epsilon)$ such that $|f(x,y) - L| < \epsilon$ holds whenever (x,y) is in the domain of f and satisfies

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta.$$

Limits and Continuity

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For example, if $\lim_{(x,y)\to(a,b)} f(x,y) = L$, $\lim_{(x,y)\to(a,b)} g(x,y) = M$, and every neighbourhood of (a,b) contains points in $\mathcal{D}(f) \cap \mathcal{D}(g)$ other than (a,b), then

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Also, if $F(t)$ is continuous at $t = L$, then

$$\lim_{(x,y)\to(a,b)} F(f(x,y)) = F(L).$$

EXAMPLE

Evaluate the following limits:

(a) $\lim_{(x,y)\to(2,3)} \left(2x - y^2\right)$

(b) $\lim_{(x,y)\to(a,b)} x^2 y$

(c) $\lim_{(x,y)\to(\pi/3,2)} y \sin\left(\frac{x}{y}\right)$

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- Along the line $y = x$ (where $x \neq 0$):

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Since we obtain different limits when approaching $(0,0)$ along different paths, the limit

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} \qquad \textbf{does not exist}.$$

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- Along any straight line $y = mx$ (where $x \neq 0$):

$$f(x,mx) = \frac{2x^2(mx)}{x^4+(mx)^2} = \frac{2mx^3}{x^4+m^2x^2} = \frac{2mx}{x^2+m^2}$$

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Since different paths lead to different limits, the limit does not exist.

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Show that the function $f(x, y) = \dfrac{x^2 y}{x^2 + y^2}$ does have a limit at the origin; specifically,

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$$|f(x,y) - 0| = \left| \frac{x^2 y}{x^2 + y^2} \right| = \frac{x^2}{x^2 + y^2} |y| \leq |y| \leq \sqrt{x^2 + y^2},$$

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which approaches zero as $(x,y) \to (0,0)$.

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It remains true that sums, differences, products, quotients, and compositions of continuous functions are continuous.

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Of course, we can *make* f continuous at $(0,0)$ by redefining its value at that point to be 0.

Partial Derivatives

Partial Derivatives

The **first partial derivatives** of the function $f(x, y)$ **with respect to the variables** x and y are the functions $f_1(x, y)$ and $f_2(x, y)$ given by

$$f_1(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h},$$

$$f_2(x,y) = \lim_{k \to 0} \frac{f(x,y+k) - f(x,y)}{k},$$

provided these limits exist.

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If $f(x, y) = x^2 \sin y$, then

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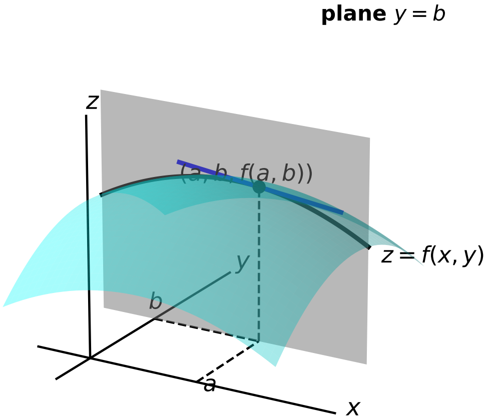

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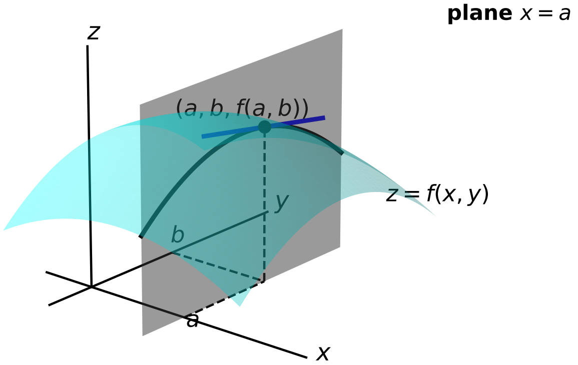

$f_2(a, b)$ is the slope of the curve of intersection of $z = f(x, y)$ and the vertical plane $x = a$ at $y = b$.

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Notations for first partial derivatives

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Values of partial derivatives

$$\left.\frac{\partial z}{\partial x}\right|_{(a,b)} = \left.\left(\frac{\partial}{\partial x} f(x, y)\right)\right|_{(a,b)} = f_1(a, b) = D_1 f(a, b)$$

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$$f_1(0, \pi) = \pi e^0 \cos(\pi) - e^0 \sin(\pi) = -\pi.$$

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The single-variable version of the Chain Rule also continues to apply to, say, $f(g(x,y))$, where f is a function of only one variable having derivative f':

$$\frac{\partial}{\partial x} f(g(x,y)) = f'(g(x,y))g_1(x,y), \qquad \frac{\partial}{\partial y} f(g(x,y)) = f'(g(x,y))g_2(x,y).$$

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Hence,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = f'\left(\frac{x}{y}\right)\left(x \times \frac{1}{y} + y \times \frac{-x}{y^2}\right) = 0.$$

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EXAMPLE

$$\frac{\partial}{\partial z}\left(\frac{2xy}{1+xz+yz}\right) = -\frac{2xy}{(1+xz+yz)^2}(x+y).$$

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Definition of partial derivatives of functions of two variables can be extended to cover functions of more than two variables.

If f is a function of n variables $x_1, x_2, \dots, x_n$, then f has n first partial derivatives, $f_1(x_1, x_2, \dots, x_n)$, $f_2(x_1, x_2, \dots, x_n)$, $\dots$, $f_n(x_1, x_2, \dots, x_n)$, one with respect to each variable.

EXAMPLE

$$\frac{\partial}{\partial z}\left(\frac{2xy}{1+xz+yz}\right) = -\frac{2xy}{(1+xz+yz)^2}(x+y).$$

Again, all the standard differentiation rules are applied to calculate partial derivatives.

Partial Derivatives

Remark If a single-variable function $f(x)$ has a derivative $f'(a)$ at $x = a$, then f is necessarily continuous at $x = a$. This property does *not* extend to partial derivatives. Even if all the first partial derivatives of a function of several variables exist at a point, the function may still fail to be continuous at that point.

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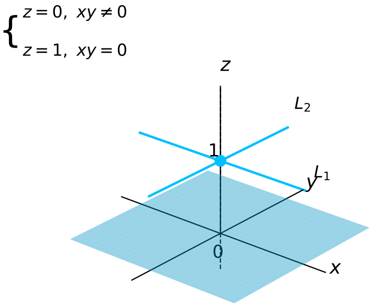

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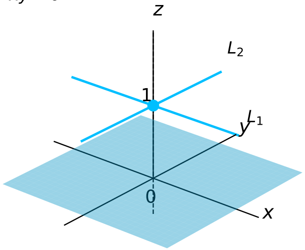
$$\begin{cases} z = 0, & xy \neq 0 \\ z = 1, & xy = 0 \end{cases}$$



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$$f(0,0) = 1$$

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Tangent Planes and Normal Lines

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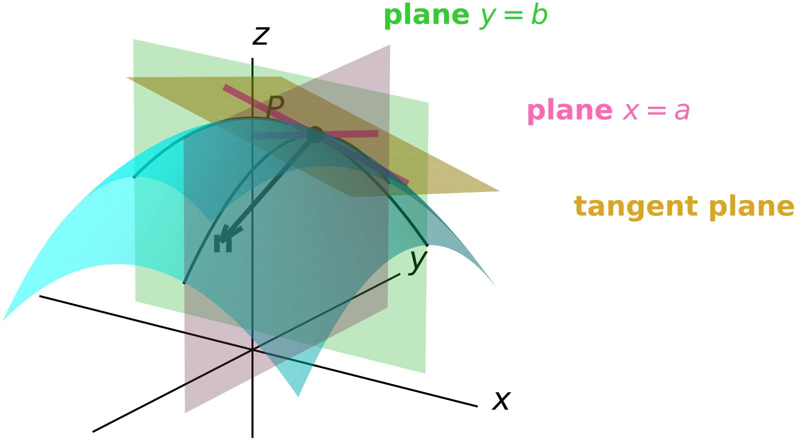

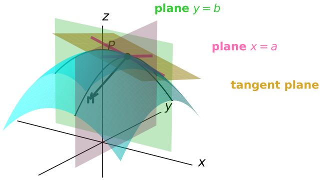

Direction of tangent on $y = b$: $\mathbf{i} + f_1(a,b)\mathbf{k}$
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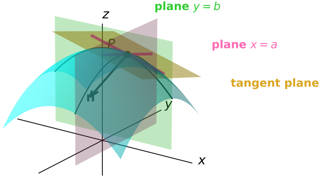

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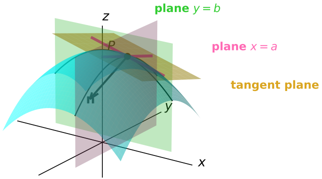

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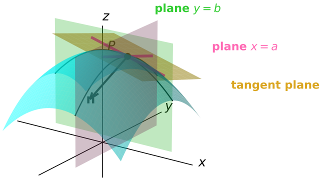

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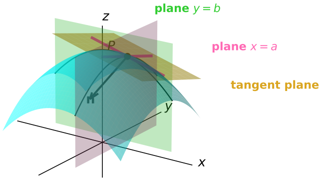

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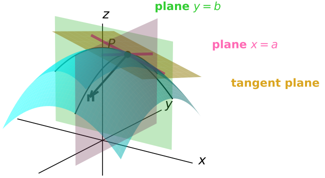

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Tangent Planes and Normal Lines

The normal line to $z = f(x, y)$ at $(a, b, f(a, b))$ has direction vector $f_1(a, b)\mathbf{i} + f_2(a, b)\mathbf{j} - \mathbf{k}$ and so has equations

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with suitable modifications if either $f_1(a, b) = 0$ or $f_2(a, b) = 0$.

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or, more simply, $3x - \pi y + 6z = 2\pi - 3\sqrt{3}$.

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$$\frac{6x - 2\pi}{-3} = \frac{6y + 6}{\pi} = \frac{6z + 3\sqrt{3}}{-6}.$$

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For these values we have $z = -31$, so the required tangent plane has equation $z = -31$ and the point of tangency is $(-4, 1, -31)$.

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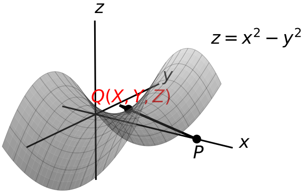

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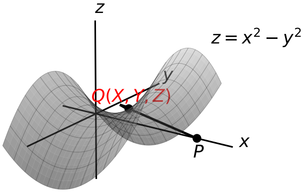

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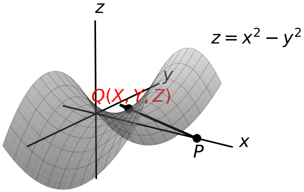

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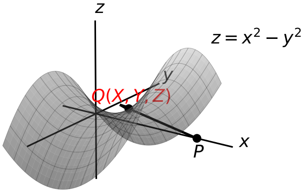

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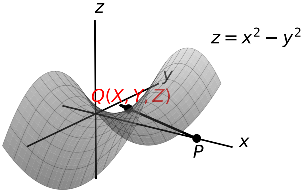

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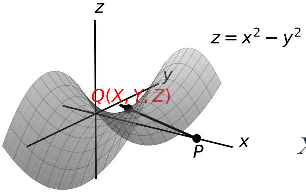

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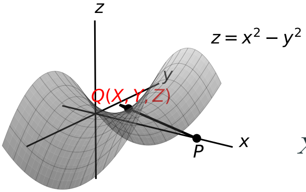

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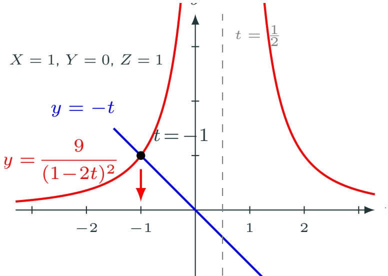

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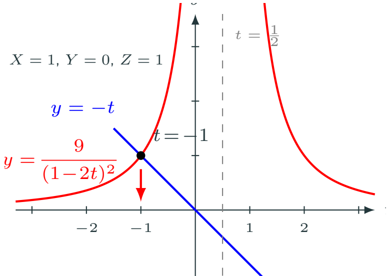

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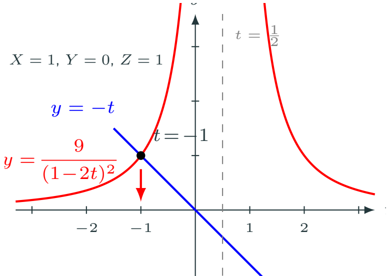

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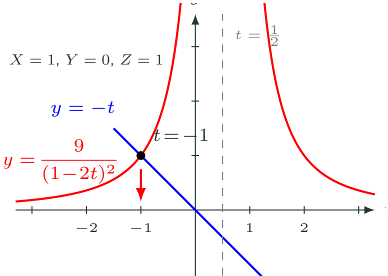

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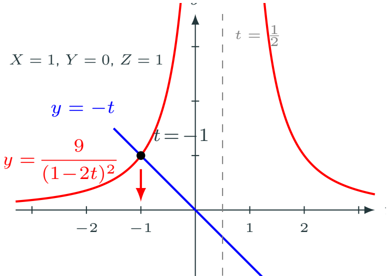

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Higher-Order Derivatives

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Similarly, if $w = f(x, y, z)$, then

$$\frac{\partial^5 w}{\partial y\,\partial x\,\partial y^2\,\partial z} = \frac{\partial}{\partial y}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial y}\frac{\partial w}{\partial z} = f_{32212}(x, y, z) = f_{zyyxy}(x, y, z).$$

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Find the four second partial derivatives of $f(x, y) = x^3 y^4$.

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Higher-Order Derivatives

EXAMPLE

Calculate $f_{223}(x, y, z)$, $f_{232}(x, y, z)$, and $f_{322}(x, y, z)$ for the function $f(x, y, z) = e^{x-2y+3z}$.

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Solution:

$$\begin{aligned} f_{223}(x,y,z) &= \frac{\partial}{\partial z}\,\frac{\partial}{\partial y}\,\frac{\partial}{\partial y}\,e^{x-2y+3z} \\ &= \frac{\partial}{\partial z}\,\frac{\partial}{\partial y}\left(-2e^{x-2y+3z}\right) \\ &= \frac{\partial}{\partial z}\left(4e^{x-2y+3z}\right) \\ &= 12\,e^{x-2y+3z}, \end{aligned}$$

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Calculate $f_{223}(x, y, z)$, $f_{232}(x, y, z)$, and $f_{322}(x, y, z)$ for the function $f(x, y, z) = e^{x-2y+3z}$.

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THEOREM Equality of mixed partials
Suppose that two mixed nth-order partial derivatives of a function f involve the same differentiations but in different orders. If those partials are continuous at a point P, and if f and all partials of f of order less than n are continuous in a neighbourhood of P, then the two mixed partials are equal at the point P.